Assignment #6
Due on Friday, February 19, 2010

Read Chapter 4 on the continuous approach to modeling bacterial growth, p. 29, in the class lecture notes webpage at http://pages.pomona.edu/~ajr04747

Do the following problems

1. Suppose the growth of a population is governed by the differential equation

\[
\frac{dN}{dt} = -kN
\]

where \( k \) is a positive constant.

(a) Explain why this model predicts that the population will decrease as time increases.

(b) If the population at \( t = 0 \) is \( N_0 \), find the time \( t \), in terms of \( k \), at which the population will be reduced by half.

2. Consider a bacterial population whose relative growth rate is given by

\[
\frac{1}{N} \frac{dN}{dt} = K
\]

where \( K = K(t) \) is a continuous function of time, \( t \).

(a) Suppose that \( N_0 = N(0) \) is the initial population density. Verify that

\[
N(t) = N_0 \exp \left( \int_0^t K(\tau) \, d\tau \right)
\]

solves the differential equation and satisfies the initial condition.

(b) Find \( N(t) \) if

\[
K(t) = \begin{cases} 
1 - t & \text{if } 0 \leq t \leq 1; \\
0 & \text{if } t > 1.
\end{cases}
\]

Sketch the graph of \( N(t) \)
3. For any population (ignoring migration, harvesting, or predation) one can model the relative growth rate by the following conservation principle
\[
\frac{dN}{dt} = \text{birth rate (per capita)} - \text{death rate (per capita)} = b - d,
\]
where \( b \) and \( d \) could be functions of time and the population density \( N \).

(a) Suppose that \( b \) and \( d \) are linear functions of \( N \) given by \( b = b_o - \alpha N \) and \( d = d_o + \beta N \) where \( b_o, d_o, \alpha \) and \( \beta \) are positive constants. Assume that \( b_o > d_o \). Sketch the graphs of \( b \) and \( d \) as functions of \( N \). Give a possible interpretation for these graphs.

(b) Find the point where the two lines sketched in part (a) intersect. Let \( K \) denote the first coordinate of the point of intersection. Show that
\[
K = \frac{b_o - d_o}{\alpha + \beta}.
\]
\( K \) is the carrying capacity of the population.

(c) Show that
\[
\frac{dN}{dt} = r N \left( 1 - \frac{N}{K} \right)
\]
where \( r = b_o - d_o \) is the intrinsic growth rate.

4. The following equation models the evolution of a population that is being harvested at a constant rate:
\[
\frac{dN}{dt} = 2N - 0.01N^2 - 75.
\]
Find equilibrium solutions and sketch a few possible solution curves. According to model, what will happen if at time \( t = 0 \) the initial population densities are 40, 60, 150, or 170?

5. Consider the modified logistic model
\[
\frac{dN}{dt} = r N \left( \frac{N}{K} \right) \left( \frac{N}{T} - 1 \right)
\]
where \( N(t) \) denotes the population density at time \( t \), and \( 0 < T < K \).

(a) Find the equilibrium solutions and determine the nature of their stability.

(b) Sketch other possible solutions to the equation.

(c) Describe what the model predicts about the population and give a possible explanation.