Solutions to Part I of Exam 1

1. Consider the difference equation $\Delta N = aN$, where a is a nonzero parameter.

   (a) Give an interpretation of the equation as a model for population growth.

   Answer: This model assumes that the per–capita growth rate of the population is constant; equivalently, the change in population in a unit of time is proportional to the population density. □

   (b) Solve the equation given that N_o is known.

   Solution: Write the equation in the form

   $$N_{t+1} = (1+a)N_t$$

   to obtain that

   $$N_t = (1+a)^t N_o \quad \text{for} \quad t = 0, 1, 2, \ldots$$

   □

   (c) Find equilibrium point(s) and test for stability. Which values of a yield stability?

   Solution: The only equilibrium point is $N^* = 0$ since $a \neq 0$. It is stable if $|1+a| < 1$, or $-2 < a < 0$. □

2. The following equation models the evolution of a population that is being harvested at a constant rate:

   $$\frac{dN}{dt} = 2N\left(1 - \frac{N}{200}\right) - 75$$

   (a) Give an interpretation for the model.

   Solution: The equation models a population that grows logistically, with intrinsic growth rate $r = 2$ and carrying capacity $K = 200$, which is also being harvested at a constant rate of 75 units of population per unit of time. □

   (b) Find equilibrium points, determine the nature of their stability, and sketch a few possible solution curves.

Solution: Write

$$
\begin{aligned}
g(N) &= 2N\left(1-\frac{N}{200}\right)-75\\
&= -\frac{1}{100}\left(N^2-200N+7500\right)\\
&= -\frac{1}{100}(N-50)(N-150).
\end{aligned}
$$

We then see that equilibrium points of the equation are

$$N_1^* = 50 \quad \text{and} \quad N_2^* = 150.$$

To determine the nature of the stability of the equilibrium points, consider the graph of g in Figure 1.

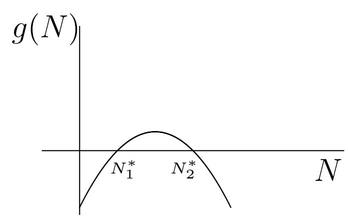

Figure 1: Graph of $g(N)$

From the information on the sign of $g(N)$ in the graph in Figure 1 we can sketch possible solutions shown in Figure 2. The sketch

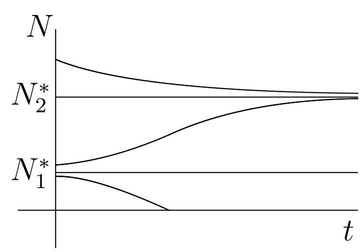

Figure 2: Possible Solutions

in Figure 2 suggests that N_1^* is unstable and N_2^* is asymptotically stable. $\square$

(c) According to model, what will happen if at time $t = 0$ the initial population density is 47? What do you conclude?

Solution: According to Figure 2, since $47 < N_1^*$, the population will go extinct in finite time. This is due to over–harvesting. $\square$