Solutions to Part II of Exam 1

3. Consider the linear first order differential equation

\[ \frac{du}{dt} = au + b, \]

where \( a \) and \( b \) are real parameters with \( a \neq 0 \).

(a) Find the equilibrium points of the equation.

**Solution:** Solve the equation \( au + b = 0 \) to get that

\[ \bar{u} = -\frac{b}{a} \]

is the only equilibrium point since \( a \neq 0 \). \( \square \)

(b) Sketch some possible solutions to the equation for the cases \( a < 0 \) and \( a > 0 \) in separate graphs. Which one of these yields stability?

**Solution:** Suppose first that \( a > 0 \), and write

\[ g(u) = au + b = a(u - \bar{u}), \]

where \( \bar{u} = -\frac{b}{a} \) is the equilibrium point found in the previous part. Since \( a > 0 \), we see that \( u'(t) > 0 \) if \( u > \bar{u} \) and \( u'(t) < 0 \) if \( u < \bar{u} \). Thus, \( u(t) \) increases for \( u > \bar{u} \) and decreases for \( u < \bar{u} \).

To get an idea of what the concavity of the graphs of solutions is, compute

\[ u''(t) = \frac{d}{dt}(u'(t)) \]

\[ = \frac{d}{dt}(g(u)) \]

\[ = g'(u)\frac{du}{dt} \]

\[ = a^2(u - \bar{u}). \]

Thus, we see that the graph of \( u = u(t) \) is concave up for \( u > \bar{u} \) and concave down is \( u < \bar{u} \). Putting all the information obtained from the signs of \( u'(t) \) and \( u''(t) \) together, we obtain the sketch shown in Figure 1.
Next, consider the case \( a < 0 \), so that \( \overline{u} > 0 \). In this case, using

\[ u'(t) = a(u - \overline{u}) \]

and

\[ u''(t) = a^2(u - \overline{u}), \]

we see that \( u(t) \) decreases for \( u > \overline{u} \) and increases for \( u < \overline{u} \); the graph of \( u = u(t) \) is concave down for \( u < \overline{u} \) and concave up for \( u > \overline{u} \). A sketch of possible solutions is shown in Figure 2. The

![Figure 2: Possible Solutions for \( a < 0 \) and \( b > 0 \)](image)

sketch in Figure 2 suggests that \( \overline{u} \) is stable for the case \( a < 0 \). □

(c) Use separation of variables to obtain solutions to the equation.

**Solution:** Write the equation in the form \( \frac{du}{dt} = a(u - \overline{u}) \), where \( \overline{u} = -\frac{b}{a} \), and separate variables to get

\[
\int \frac{1}{u - \overline{u}} \, du = \int a \, dt,
\]

which yields

\[
\ln |u - \overline{u}| = at + c_1,
\]
for some constant $c_1$. Exponentiating on both sides of the previous equation, and then solving for $u = u(t)$ yields

$$u(t) = \bar{u} + C e^{at},$$

for some constant $C$. □

(d) Use your result from the previous part to justify your answers to part (b).

**Solution:** If $a < 0$, it follows from the result in equation (1) that

$$\lim_{t \to \infty} u(t) = \bar{u}.$$ 

Thus, $\bar{u}$ is asymptotically stable in this case.

We also get from (1) that

$$|u(t) - \bar{u}| = |C| e^{at}$$

for all $t \in \mathbb{R}$. Thus, if $a > 0$, the distance from $u(t)$ to the equilibrium point, $\bar{u}$, increases as $t$ increases. Hence, if $a > 0$, then $\bar{u}$ is unstable. □