Solutions to Review Problems for Exam #1

1. Consider the difference equation $X_{t+1} = \lambda X_t + a$, where λ and a are real parameters, given that X_0 is known.

   (a) Find a closed form solution, X_t, to the equation and discuss how the behavior of the solution as $t \to \infty$ is determined by the value of λ.

   Solution: We may proceed by induction to find a formula for X_n. Starting with the base case:

$$X_1 = \lambda X_o + a,$$

   and going onto the subsequent cases, we find that

$$\begin{aligned} X_2 &= \lambda X_2 + a \\ &= \lambda(\lambda X_o + a) + a \\ &= \lambda^2 X_o + \lambda a + a, \end{aligned}$$

   and

$$\begin{aligned} X_3 &= \lambda X_2 2 + a \\ &= \lambda(\lambda^2 X_o + \lambda a + a) + a \\ &= \lambda^3 X_o + \lambda^2 a + \lambda a + a. \end{aligned}$$

   These cases suggest that

$$X_n = \lambda^n X_o + (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1)a, \tag{1}$$

   which may be shown by induction on n.
   If $\lambda \neq 1$, we may write (1) as

$$X_n = \lambda^n X_o + \frac{\lambda^n - 1}{\lambda - 1}\, a. \tag{2}$$

   We consider first the case in which $\lambda = 1$. In this case, it follows from (1) that

$$X_n = X_o + na;$$

   Thus, if $a > 0$, then $X_n \to +\infty$ as $n \to \infty$; if $a < 0$, then $X_n \to -\infty$ as $n \to \infty$; and if $a = 0$, then X_n is constant.
   If $\lambda \neq 1$, we consider the following two cases separately: (i) $0 < \lambda < 1$ and (ii) $\lambda > 1$.

(i) If $0 < \lambda < 1$, it follows from (2) that

$$\lim_{n\to\infty} X_n = \frac{a}{1-\lambda},$$

since $\lim_{n\to\infty} \lambda^n = 0$ in the case $0 < \lambda < 1$.

(ii) If $\lambda > 1$, we see that $X_n \to \infty$ as $n \to \infty$ if $a > 0$ and $X_o > 0$. If both X_o and a are negative, then $X_n \to -\infty$ as $n \to \infty$.

□

(b) Write the difference equation in the form $X_{t+1} = f(X_t)$, for some function f.
Give the equilibrium point(s) of the equation and use the principle of linearized stability to determine the nature of their stability.

***Solution**:* The function f is given by

$$f(x) = \lambda x + a \quad \text{ for all } x \in \mathbf{R}.$$

To find equilibrium points, we solve the equation

$$f(x) = x,$$

that is, we find the fixed–points of f. Solving

$$\lambda x + a = x,$$

we find that

$$x = \frac{a}{1-\lambda}$$

if $\lambda \neq 1$.
The derivative of f is $f'(x) = \lambda$. Thus, the principle of linearized stability implies that $x^* = \dfrac{a}{1-\lambda}$ is asymptotically stable if $|\lambda| < 1$, and unstable if $|\lambda| > 1$. If $|\lambda| = 1$, the principle of linearized stability does not apply. □

2. Find the equilibrium point of the difference equation $X_{t+1} = X_t^2 - 6$, and determine their stability properties.

***Solution**:* Let $f(x) = x^2 - 6$. The fixed–points of the difference equation are solutions of

$$f(x) = x,$$

or

$$x^2 - 6 = x,$$

which yields $x_1^* = -2$ and $x_2^* = 3$. In order to determine the stability of the fixed–points, we compute

$$f'(x) = 2x.$$

Observe that $|f'(x_1^*)| = 4$ and $|f'(x_2^*)| = 6$. In both cases we get that $|f'(x^*)| > 1$. Thus, by the principled of linearized stability, both fixed–points are unstable. □

3. Suppose the growth of a population of size N_t at time t is dictated by the discrete model

$$N_{t+1} = \frac{400N_t}{(10 + N_t)^2}.$$

(a) Find the biologically reasonable fixed points for this difference equation.

***Solution**:* Let $f(x) = \dfrac{400x}{(10+x)^2}$ for $x \in \mathbf{R}$. Then, the fixed points of the equation are solutions of

$$f(x) = x,$$

or

$$\frac{400x}{(10+x)^2} = x.$$

To solve this equation, first write

$$\frac{400x}{(10+x)^2} - x = 0,$$

or

$$x\left(\frac{400}{(10+x)^2} - 1\right) = 0,$$

from which we get that

$$x = 0 \quad \text{or} \quad (10+x)^2 = 400.$$

We therefore get that

$$x = 0, \text{ or } x = -30, \text{ or } \ x = 10.$$

Out of these fixed points, only the first and the last are biologically reasonable. Hence, $N_1^* = 0$ and $N_2^* = 10$. □

(b) Determine the stability properties of the equilibrium points found in the previous part.

Solution: Compute the derivative of f to get

$$f'(x) = \frac{400(10-x)}{(10+x)^3}.$$

Then

$$f'(N_1^*) = \frac{400(10)}{(10)^3} = 4 > 1,$$

and therefore $N_1^* = 0$ is unstable.
On the other hand, since

$$|f'(N_2^*)| = 0 < 1,$$

$N_2^* = 10$ is stable. □

(c) If $N_0 = 5$, what happens to the population in the long run?

Answer: $\lim\limits_{t\to\infty} N_t = 10.$ □

4. We have seen that the (continuous) logistic model $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$, where r and K are positive parameters, has an equilibrium point at $\overline{N} = K$.

(a) Let $g(N) = rN\left(1 - \dfrac{N}{K}\right)$ and give the linear approximation to $g(N)$ for N close to K:

$$g(K) + g'(K)(N-K).$$

Observe that $g(K) = 0$ since K is an equilibrium point.

Solution: Compute the derivative $g'(N)$ to get

$$g'(N) = r - \frac{2r}{K}N.$$

Then,

$$g'(K) = r - \frac{2r}{K}K = -r$$

and therefore the linear approximation to $g(N)$ for N near K is

$$-r(N-K).$$

□

(b) Let $u = N - K$ and consider the linear differential equation

$$\frac{du}{dt} = g'(K)u.$$

This is called the *linearization* of the equation

$$\frac{dN}{dt} = g(N)$$

around the equilibrium point $\overline{N} = K$.
Use separation of variables to solve this equation. What happens to $|u(t)|$ as $t \to \infty$, where u is any solution to the linearized equation?

Solution: Solve the equation

$$\frac{du}{dt} = -ru$$

to obtain that

$$u(t) = ce^{-rt}$$

for some constant c. Then,

$$|u(t)| = |c|e^{-rt}$$

and therefore

$$\lim_{t\to\infty} |u(t)| = 0.$$

□

(c) Use your result in the previous part to give an explanation as to why any solution to the logistic equation that begins very close to K can be approximation by $K+u(t)$, where u is a solution to the linearized equation.

Solution: Let $N(t)$ denote a solution to the logistic equation with $N(0) = N_o$ and N_o very close to K. Then $|u(0)| = |N_o - K|$ is very small and consequently,

$$|u(t)| = |N_o - K|e^{-rt} < |N_o - K| \quad \text{for all} \quad t > 0.$$

Thus, $|u(t)|$ if very small for all $t > 0$ and therefore the function $g(N(t))$ is very close to its linear approximation

$$-r(N - K) = -ru.$$

Consequently, a solution of $\dfrac{dN}{dt} = g(N)$ can be approximated by a solution of

$$\frac{dN}{dt} = -r(N - K),$$

or

$$\frac{du}{dt} = -ru.$$

Thus, $N(t) - K$ can be approximated by $u(t)$ for all $t > 0$, and therefore

$$N(t) \approx K + u(t).$$

□

(d) Suppose that $N = N(t)$ is a solution to the logistic equation that starts at N_o, where N_o is very close to K. Find an estimate of the time it takes for the distance $|N(t) - K|$ to decrease by a factor of e. This time is called the *recovery time.*

Solution: Since

$$N(t) \approx K + u(t),$$

$$N(t) \approx K + (N_o - K)e^{-rt}.$$

for all $t > 0$. So,

$$|N(t) - K| \approx |N_o - K|e^{-rt}$$

for all $t > 0$.

We want to know the time t for which

$$|N(t) - K| = \frac{|N_o - K|}{e}.$$

This is approximated by the time t for which

$$|N_o - K|e^{-rt} = \frac{|N_o - K|}{e}$$

or

$$e^{-rt} = e^{-1}$$

This yields that $rt = 1$, or $t = 1/r$. □

5. *[Harvesting]* The following differential equation models the growth of a population of size $N = N(t)$ that is being harvested at a rate proportional to the population density

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN, \tag{3}$$

where r, K and E are parameters and non–negative parameters with $r > 0$ and $K > 0$.

(a) Give an interpretation for this model. In particular, give interpretation for the term EN. The parameter E is usually called the harvesting *effort.*

Answer: This equation models a population that grows logistically and that is also being harvested at a rate proportional to the populations density. □

(b) Calculate the equilibrium points for the equation (3), and give conditions on the parameters that yield a biologically meaningful equilibrium point. Determine the nature of the stability of that equilibrium point. Sketch possible solutions to the equation in this situation.

Solution: Write

$$\begin{aligned} g(N) &= rN\left(1 - \frac{N}{K}\right) - EN \\ &= rN\left(1 - \frac{N}{K} - \frac{E}{r}\right) \\ &= -\frac{r}{K}N\left[N - K\left(1 - \frac{E}{r}\right)\right]. \end{aligned}$$

We then see that equilibrium points of equation (3) are

$$N_1^* = 0 \quad \text{and} \quad N_2^* = K\left(1 - \frac{E}{r}\right).$$

The second equilibrium point is biologically meaningful if $N_2^* > 0$, and for this to happen we require that $E < r$; that is, the harvesting effort is less than the intrinsic growth rate.

To determine the nature of the stability of N_2^* for the case $E < r$, consider the graph of g in Figure 1. Observe from the graph that $g'(N_2^*) < 0$. It then follows from the principle of linearized stability that N_2^* is asymptotically stable.

The solid curves in Figure 2 show some possible solutions of the equation □

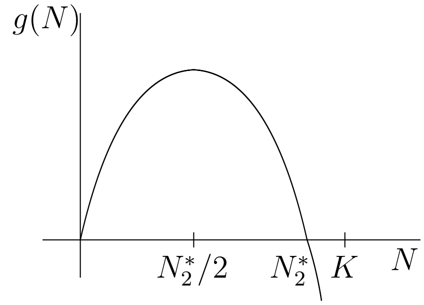

Figure 1: Graph of $g(N)$

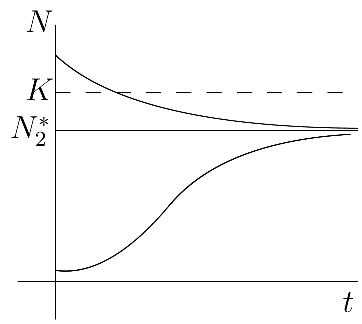

Figure 2: Possible Solutions

(c) What does the model predict if $E \geq r$?

Solution: If $E = r$, then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{r}{K}N^2 < 0$$

for $N > 0$. It then follows that $N(t)$ will always be strictly decreasing and so the population will go extinct. In fact, using separation of variables, we obtain that the solution for $N(0) = N_o$ is given by

$$N(t) = \frac{N_o K}{K + N_o r t},$$

which tends to 0 as $t \to \infty$.

On the other hand, if $E > r$, then

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= -\frac{r}{K}N\left[N - K\left(1 - \frac{E}{r}\right)\right] \\ &= -\frac{r}{K}N^2 + KN(r - E) \\ &< -\frac{r}{K}N^2 < 0, \end{aligned}$$

and so again we conclude the $N(t)$ will be always decreasing to 0.
□

6. *[Harvesting, continued]* Suppose that $0 < E < r$ in equation (3), and let $\overline{N}$ denote the positive equilibrium point. The quantity $Y = E\overline{N}$ is called the *harvesting yield.*

   (a) Find the value of E for which the harvesting yield is the largest possible; this value of the yield is called the *maximum sustainable yield.*

   Solution: $\overline{N}$ is N_2^* in the previous problem. Consequently, the yield is given by

   $$Y(E) = EN_2^* = EK\left(1 - \frac{E}{r}\right) = EK - \frac{K}{r}E^2.$$

   Taking derivatives with respect to E, we obtain that

   $$Y'(E) = K - \frac{2K}{r}E \quad \text{and} \quad Y''(E) = -\frac{2K}{r} < 0.$$

   Thus, by the second derivative test, $Y(E)$ has a maximum when $E = \dfrac{r}{2}$. □

   (b) What is the value of the equilibrium point for which there is the maximum sustainable yield?

   Solution: The maximum value of Y is

   $$Y(r/2) = \frac{r}{2}K\left(1 - \frac{r/2}{r}\right) = \frac{rK}{4}.$$

   □