Solutions to Part I of Exam 2

1. Suppose that the rate at which a drug leaves the bloodstream and passes into the urine at a given time is proportional to the quantity of the drug in the blood at that time.

   (a) Write down and solve a differential equation for the quantity, $Q = Q(t)$, of the drug in the blood at time, $t$, in hours. State all the assumptions you make and define all the parameters that you introduce.

   Solution: By the conservation principle for a one–compartment model,
   \[
   \frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out},
   \]
   where 
   \[
   \text{Rate of } Q \text{ in} = 0
   \]
   and 
   \[
   \text{Rate of } Q \text{ out} = kQ,
   \]
   for some constant of proportionality $k$. Thus, $Q$ satisfies the differential equation
   \[
   \frac{dQ}{dt} = -kQ,
   \]
   which has solution
   \[
   Q(t) = ce^{-kt} \quad \text{for all } t \geq 0.
   \]
   for some constant $c$. □

   (b) Suppose that an initial dose of $Q_o$ is injected directly into the blood, and that 20% of that initial amount is is left in the blood after 3 hours. Based on the solution you found in the previous part, write down $Q(t)$ for this situation and sketch its graph.

   Solution: If $Q(0) = Q_o$, then $c = Q_o$. Thus,
   \[
   Q(t) = Q_o e^{-kt} \quad \text{for all } t \geq 0.
   \]
   If $Q(3) = 0.2Q_o$, then
   \[
   0.2Q_o = Q_o e^{-3k},
   \]
from which we obtain that 
\[ k = -\frac{1}{3} \ln(0.2) \approx 0.54. \]

It then follows that 
\[ Q(t) = Q_0 e^{\frac{k}{3} \ln(0.2)} \approx Q_0 e^{-0.54t}. \]

\[ \square \]

Figure 1: Sketch of graph of \( Q(t) \)

(c) How much of the drug is left in the patient’s body after 6 hours if the patient is given 100 mg initially?

**Solution:** Compute

\[
Q(6) = 100e^{\frac{6}{3} \ln(0.2)} = 100e^{2\ln(0.2)} = 100(0.2)^2 = \frac{100}{25} = 4.
\]

Thus, there will be 4 mg of the drug left in the patient after 6 hours. \[ \square \]

2. Suppose a bacterial colony has \( N_o \) bacteria at time \( t = 0 \). Let \( M(t) \) denote the number of bacteria that develop certain mutation during the time interval \([0, t]\). Assume that, for small \( \Delta t > 0 \),

\[
M(t + \Delta t) - M(t) \approx a (\Delta t) N(t), \quad (1)
\]
where \( a \) is a positive constant, and \( N(t) \) is the number of bacteria in the colony at time \( t \).

(a) Give an interpretation to what the expression in (1) is saying. In particular, provide a meaning for the constant, \( a \), known as the mutation rate.

**Solution:** The expression in (1) postulates that the number of mutations occurring in the time interval \([t, t + \Delta t]\) is proportional to the length of the interval, \( \Delta t \), and the number of cells, \( N(t) \), present at time \( t \). The constant of proportionality, \( a \), can be interpreted as the fraction of cells that mutate in a unit of time.

(b) Let \( \mu(t) = E(M(t)) \) denote the expected value of the number of mutations in the time interval \([0, t]\). It is possible to prove, using the expression in (1), that \( \mu = \mu(t) \) is differentiable and satisfies the differential equation

\[
\frac{d\mu}{dt} = aN(t).
\]

Solve the differential equation in (2) assuming that \( N(t) \) grows in time according to a Malthusian model with per–capita growth rate \( k \), and that there are no mutant bacteria at time \( t = 0 \).

**Solution:** Assuming that the bacterial colony is growing according the Malthusian model

\[
\begin{align*}
\frac{dN}{dt} &= kN \\
N(0) &= N_o,
\end{align*}
\]

where \( k = \frac{\ln 2}{T} \), \( T \) being the doubling time or the duration of a division cycle, then \( N(t) = N_o e^{kt} \). Substituting this into (2) we get

\[
\frac{d\mu}{dt} = aN_o e^{kt},
\]

which can be integrated to yield

\[
\mu(t) - \mu(0) = \int_0^t aN_o e^{k\tau} \, d\tau = \frac{a}{k} N_o (e^{kt} - 1).
\]
If there no mutations at time $t = 0$, $\mu(0) = 0$, and so

$$\mu(t) = \frac{a}{k}(N_0 e^{kt} - N_0),$$

or

$$\mu(t) = \frac{a}{k}(N(t) - N_0).$$

Hence, the average number of mutations which occur in the interval $[0, t]$ is proportional to the population increment during that time period. The constant of proportionality is the mutation rate divided by the growth rate. \qed