Assignment #16

Due on Friday, April 15, 2011

Read Section 5.4 on *The Fundamental Theorem of Calculus,* pp. 292–295, in Baxandall and Liebek's text.

Read Section 5.5 on *Potential Functions and Conservative Fields,* pp. 296–308, in Baxandall and Liebek's text.

Read Section 5.2 on *Line Integrals* in the class Lecture Notes (pp. 69–72).

Background and Definitions

- *(Path Connected Sets)* A set $U \subseteq \mathbb{R}^n$ is said to be path connected if and only if for any vectors p and q in U, there exists a C^1 path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = p$, $\sigma(1) = q$ and $\sigma(t) \in U$ for all $t \in [0,1]$; i.e., any two elements in U can be connected by a C^1 path whose image is entirely contained in U.

- *(Flux Across a Simple, Closed Curve in $\mathbb{R}^2$)* Let U denote an open subset of $\mathbb{R}^2$ and $F\colon U \to \mathbb{R}^2$ be a two–dimensional vector field given by

$$F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j}, \qquad \text{for all } (x,y) \in U,$$

where P and Q are scalar fields defined in U. Let C denote a simple, piece–wise C^1, closed curve contained in U, which is oriented in the counterclockwise sense. The flux of F across C, denoted by $\oint_C F \cdot \widehat{n}\; \mathrm{d}s$, is defined by

$$\oint_C F \cdot \widehat{n}\; \mathrm{d}s = \int_C P(x,y)\; \mathrm{d}y - Q(x,y)\; \mathrm{d}x,$$

where $\widehat{n}$ denotes the outward unit normal to the curve C, wherever it is defined.

Do the following problems

1. Integrate the 1–form $yz\; \mathrm{d}x + xz\; \mathrm{d}y + xy\; \mathrm{d}z$ over each of the following curves in $\mathbb{R}^3$ which connect $(0,1,0)$ to $(2,1,1)$.

   (a) the straight line from $(0,1,0)$ to $(2,1,1)$,

   (b) the lines from $(0,1,0)$ to $(0,1,1)$ to $(2,1,1)$,

   (c) the lines from $(0,1,0)$ to $(2,1,0)$ to $(2,1,1)$,

   (d) the arc $(2t, (2t-1)^2, t)$, for $0 \leqslant t \leqslant 1$.

2. Let U denote an open subset of $\mathbb{R}^n$ which is path connected, and let $F\colon U \to \mathbb{R}^n$ be a vector field with the property that

$$\int_C F \cdot \mathrm{d}\overrightarrow{r} = 0,$$

for any simple, piece–wise C^1, closed curve, C, contained in U.

Let p and q be points in U. Since U is path connected, there exists a path C^1 path, $\sigma\colon [0,1] \to U$, connecting p to q. Assume that σ parametrizes a curve C_1 in U. Prove that if $\gamma\colon [0,1] \to U$ is another C^1 path that connects p to q, and $C_2 = \gamma([0,1])$ is paramatrized by γ, then

$$\int_{C_1} F \cdot \mathrm{d}\overrightarrow{r} = \int_{C_2} F \cdot \mathrm{d}\overrightarrow{r}.$$

3. Let U denote an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^n$ be a vector field with the property that $F(v) = \nabla f(v)$ for all $v \in U$, where $f\colon U \to \mathbb{R}$ is a C^1 scalar field.

Prove that if C is any C^1, simple, closed curve in U, then

$$\int_C F \cdot \mathrm{d}\overrightarrow{r} = 0.$$

4. Let $F(x,y) = x^2\,\widehat{i} + y^2\,\widehat{j}$ and C be the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, oriented in the in the counterclockwise sense. Compute the flux of F across C.

5. Compute the flux, $\oint_C F \cdot \widehat{n}\ \mathrm{d}s$, where $F(x,y) = x\,\widehat{i} + y\,\widehat{j}$, for all $(x,y) \in \mathbb{R}^2$ and C is the unit circle oriented in the counterclockwise sense.