Assignment #17

Due on Wednesday, April 20, 2011

Read Section 11.2 on *Differential* 1–*Forms,* pp. 523–526, in Baxandall and Liebek's text.

Read Section 5.5 on *Differential Forms* in the class Lecture Notes (pp. 75–87).

Background and Definitions

- *(Differential* 0–*Forms)* A differential 0 form in an open set $U \subseteq \mathbb{R}^n$ is a C^∞ function, $f\colon U \to \mathbb{R}^n$. A differential 0–form acts on points, p, in U be means of function evaluation: $f_p = f(p)$, for all $p \in U$.

- *(Differential* 1–*Forms)* Let U denote an open subset of $\mathbb{R}^n$ and let $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ denote the space of real valued linear transformations defined in $\mathbb{R}^n$. A differential 1–form, ω, on U is a (smooth) map $\omega\colon U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ which assigns to each $p \in U$, and a linear transformation $\omega_p\colon \mathbb{R}^n \to \mathbb{R}$ given by

  $$\omega_p(h) = F_1(p)h_1 + F_2(p)h_2 + \cdots + F_n(p)h_n,$$

  for all $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$, where the vector field $F = (F_1, F_2, \ldots, F_n)$ is a smooth vector field.

  Differential 1 forms act on oriented, smooth curves, C, by means on integration; we write

  $$\omega(C) = \int_C \omega = \int_C F_1\ dx_1 + F_2\ dx_2 + \cdots + F_n\ dx_n.$$

Do the following problems

1. Evaluate the differential 1–form $\omega = \dfrac{-y}{x^2+y^2}\ dx + \dfrac{x}{x^2+y^2}\ dy$ in the directed line segment from $P_o(1,1)$ to $P_1(0,1)$.

2. A differential 1–form, ω, is said to be exact if there exists a 0–form, f, such that $\omega = df$ in the domain of definition of f and ω. Determine which of the following 1–forms are exact.

   (a) $yz\ \mathrm{d}x + xz\ \mathrm{d}y + xy\ \mathrm{d}z$

   (b) $xy\ \mathrm{d}x + yz\ \mathrm{d}y + xz\ \mathrm{d}z$

   (c) $(2xyz + z)\ \mathrm{d}x + (x^2z + 1)\ \mathrm{d}y + (x^2y + x)\ \mathrm{d}z$

3. Show that a differential 1–form

$$\omega = F_1\ dx_1 + F_2\ dx_2 + \cdots + F_n\ dx_n$$

is exact if and only if the vector field $F = F_1\ e_1 + F_2\ e_2 + \cdots + F_n\ e_n$ is the gradient of a smooth function $f\colon U \to \mathbb{R}$.

4. Let $\omega = -y\ dx + x\ dy$. Evaluate the differential 1–form on the unit circle, C, oriented in the counterclockwise sense.

5. Let ω demote a differential 1–form in $\mathbb{R}^n$, and let P_1 and P_2 be any two points in $\mathbb{R}^n$. Show that

$$\int_{[P_1,P_2]} \omega = -\int_{[P_2,P_1]} \omega.$$