Assignment #5

Due on Monday, February 7, 2011

Read Section 2.1 on *Vector–Valued Functions of* $\mathbb{R}$ in Baxandall and Liebek's text (pp. 26–29).

Read Section 4.1 on *Vector–Valued Functions of* $\mathbb{R}^m$ in Baxandall and Liebek's text (pp. 182–184).

Read Section 4.2 on *Continuity and Limits* in Baxandall and Liebek's text (pp. 185–188).

Read Section 3.1 on *Types of Functions in Euclidean Space* in the class Lecture Notes (pp. 25–26).

Read Section 3.2 on *Open Subsets of Euclidean Space* in the class Lecture Notes (pp. 26–27).

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes (pp. 27–33).

Do the following problems

1. Let U_1 and U_2 denote subsets in $\mathbb{R}^n$.

   (a) Show that if U_1 and U_2 are open subsets of $\mathbb{R}^n$, then their intersection

   $$U_1 \cap U_2 = \{y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2\}$$

   is also open.

   (b) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \right\}$ is not an open subset of $\mathbb{R}^2$.

2. In Problem 4 of Assignment #3 you proved that every linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}$ must be of the form

   $$T(v) = w \cdot v \quad \text{for every } \; v \in \mathbb{R}^n,$$

   where w is some vector in $\mathbb{R}^n$. Use this fact, together with the Cauchy–Schwarz inequality, to prove that T is continuous at every point in $\mathbb{R}^n$.

3. A subset, U, of $\mathbb{R}^n$ is said to be **convex** if given any two points x and y in U, the straight line segment connecting them is entirely contained in U; in symbols,

$$\{x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leqslant 1\} \subseteq U$$

   (a) Prove that the ball $B_r(O) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$ is a convex subset of $\mathbb{R}^n$.

   (b) Prove that the "punctured unit disc" in $\mathbb{R}^2$,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; 0 < x^2 + y^2 < 1 \right\},$$

   is not a convex set.

4. Let x and y denote real numbers.

   (a) Starting with the self–evident inequality: $(|x| - |y|)^2 \geqslant 0$, derive the inequality

$$|xy| \leqslant \frac{1}{2}(x^2 + y^2).$$

   (b) Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

   Use the inequality derived in the previous part to prove that f is continuous at the origin.

5. Let

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

   Define $f(0, 0)$ so that $f(x, y)$ is continuous at $(0, 0)$. Justify your answer.