Assignment #6

Due on Monday, February 14, 2011

Read Section 4.2 on *Continuity and Limits* in Baxandall and Liebek's text (pp. 185–188).

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes (pp. 29–35).

Background and Definitions

- *(Continuous Function)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if $\displaystyle\lim_{\|y-x\|\to 0} \|F(y) - F(x)\| = 0$.
- *(Image)* If $A \subseteq U$, the *image of* A under the map $F\colon U \to \mathbb{R}^m$, denoted by $F(A)$, is defined as the set $F(A) = \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in A\}$.
- *(Pre–image)* If $B \subseteq \mathbb{R}^m$, the *pre–image of* B under the map $F\colon U \to \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set $F^{-1}(B) = \{x \in U \mid F(x) \in B\}$.

  Note that $F^{-1}(B)$ is always defined even if F does not have an inverse map.

Do the following problems

1. Use the triangle inequality to prove that, for any x and y in $\mathbb{R}^n$,
$$\big|\|y\| - \|x\|\big| \leqslant \|y - x\|.$$
   Use this inequality to deduce that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by
$$f(x) = \|x\| \quad \text{for all } \ x \in \mathbb{R}^n$$
   is continuous on $\mathbb{R}^n$.

2. Let $f(x, y)$ and $g(x, y)$ denote two functions defined on a open region, D, in $\mathbb{R}^2$. Prove that the vector field $F\colon D \to \mathbb{R}^2$, defined by
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$
   is continuous on D if and only f and g are both continuous on D.

3. Let U denote an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^m$ and $G\colon U \to \mathbb{R}^m$ be two given functions.

   (a) Explain how the sum $F + G$ is defined.

   (b) Prove that if both F and G are continuous on U, then their sum is also continuous.
   (*Suggestion:* The triangle inequality might come in handy.)

4. In each of the following, given the function $F\colon U \to \mathbb{R}^m$ and the set B, compute the pre–image $F^{-1}(B)$.

   (a) $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, and $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

   (b) $f\colon D' \to \mathbb{R}$,

   $$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}, \quad \text{for} \quad (x, y) \in D'$$

   where $D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ (the punctured unit disc), $B = \{1\}$.

   (c) $f\colon D' \to \mathbb{R}$ is as in part (b), and $B = \{2\}$.

   (d) $f\colon D' \to \mathbb{R}$ is as in part (b), and $B = \{1/2\}$.

5. Compute the image the given sets under the following maps

   (a) $\sigma\colon \mathbb{R} \to \mathbb{R}^2$, $\sigma(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Compute $\sigma(\mathbb{R})$.

   (b) $f\colon D' \to \mathbb{R}$ and D' are as given in part (b) of the previous problem. Compute $f(D')$.