

Review Problems for Exam 2

1. Let U denote an open and convex subset of \mathbb{R}^n . Suppose that $f: U \rightarrow \mathbb{R}$ is differentiable at every $x \in U$. Fix x and y in U , and define $g: [0, 1] \rightarrow \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for } 0 \leq t \leq 1.$$

- (a) Explain why the function g is well defined.
 (b) Show that g is differentiable on $(0, 1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for } 0 < t < 1.$$

- (c) Use the Mean Value Theorem for derivatives to show that there exists a point z on the line segment connecting x to y such that

$$f(y) - f(x) = D_{\hat{u}}f(z)\|y - x\|,$$

where \hat{u} is the unit vector in the direction of the vector $y - x$; that is,

$$\hat{u} = \frac{1}{\|y - x\|}(y - x).$$

- (d) Prove that if U is an open and convex subset of \mathbb{R}^n , and $f: U \rightarrow \mathbb{R}$ is differentiable on U with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then f must be a constant function.
2. Let U be an open subset of \mathbb{R}^n and I be an open interval. Suppose that $f: U \rightarrow \mathbb{R}$ is a differentiable scalar field and $\sigma: I \rightarrow \mathbb{R}^n$ be a differentiable path whose image lies in U . Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

3. Let C denote the boundary of the oriented triangle, $T = [(0, 0)(1, 0)(1, 2)]$, in \mathbb{R}^2 . Evaluate the line integral $\int_C \frac{x^2}{2} dy - \frac{y^2}{2} dx$, by applying the Fundamental Theorem of Calculus.
4. Let $F(x, y) = 2x \hat{i} - y \hat{j}$ and R be the square in the xy -plane with vertices $(0, 0)$, $(2, -1)$, $(3, 1)$ and $(1, 2)$. Evaluate $\oint_{\partial R} F \cdot n \, ds$.

5. Evaluate the line integral $\int_{\partial R} (x^4 + y) \, dx + (2x - y^4) \, dy$, where R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 3, -2 \leq y \leq 1\},$$

and ∂R is traversed in the counterclockwise sense.

6. Integrate the function given by $f(x, y) = xy^2$ over the region, R , defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, 0 \leq y \leq 4 - x^2\}.$$

7. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (1)$$

for $a > 0$ and $b > 0$.

- (a) Evaluate the line integral $\oint_{\partial R} x \, dy - y \, dx$, where ∂R is the ellipse in (1) traversed in the positive sense.
- (b) Use your result from part (a) and the Fundamental Theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (1).
8. Evaluate the double integral $\int_R e^{-x^2} \, dx \, dy$, where R is the region in the xy -plane sketched in Figure 1.

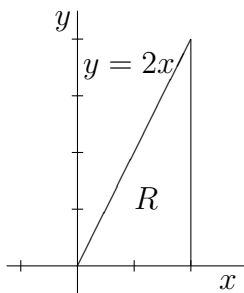


Figure 1: Sketch of Region R in Problem 8