Review Problems for Final Exam

1. In this problem, x and y denote vectors in $\mathbb{R}^n$.

   (a) Use the triangle inequality to derive the inequality
   $$|\, \|y\| - \|x\| \,| \leqslant \|y - x\| \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

   (b) Use the inequality derived in the previous part to show that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|x\|$, for all $x \in \mathbb{R}^n$, is continuous.

   (c) Prove that the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = \sin(\|x\|)$, for all$x \in \mathbb{R}^n$, is continuous.

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \|x\|^2$ for all $x \in \mathbb{R}^n$.

   (a) Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map
   $$Df(x)\colon \mathbb{R}^n \to \mathbb{R} \quad \text{for all} \quad x \in \mathbb{R}^n.$$
   What is the gradient of f at x for all $x \in \mathbb{R}^n$?

   (b) Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector v in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|v - t\widehat{u}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

   (c) Let $\widehat{u}$ be as in the previous part. For any $v \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{u}$ which is the closest to v. Justify your answer.

3. Let I denote an open interval which contains the real number a. Assume that $\sigma\colon I \to \mathbb{R}^n$ is a C^1 parametrization of a curve C in $\mathbb{R}^n$. Define $s\colon I \to \mathbb{R}$ as follows:
   $$s(t) = \text{arlength along the curve } C \text{ from } \sigma(a) \text{ to } \sigma(t),$$
   for all $t \in I$.

   (a) Give a formula, in terms of an integral, for computing $s(t)$ for all $t \in I$.

   (b) Prove that s is differentiable on I and compute $s'(t)$ for all $t \in I$.
   Deduce that s is strictly increasing with increasing t.

4. Compute the arc length along the portion of the cycloid given by the parametric equations
$$x = t - \sin t \quad \text{and} \quad y = 1 - \cos t, \quad \text{for} \ \ t \in \mathbb{R},$$
from the point $(0,0)$ to the point $(2\pi, 0)$.

5. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by
$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$
and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

   (a) Show that Φ is differentiable and give a formula for its derivative, $D\Phi(u,v)$, at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

   (b) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

   (c) Evaluate the integral $\displaystyle\iint_R dxdy$, where R is the region in the xy–plane obtained in part (b).

   (d) Evaluate the integral $\displaystyle\iint_T |\det[D\Phi(u,v)]|\ dudv$, where $\det[D\Phi(u,v)]$ denotes the determinant of the Jcobian matrix of Φ obtained in part (a). Compare the result obtained here with that obtained in part (c).

6. Consider the iterated integral $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1-y^2}\ dydx.$

   (a) Identify the region of integration, R, for this integral and sketch it.

   (b) Change the order of integration in the iterated integral and evaluate the double integral $\displaystyle\int_R x\sqrt{1-y^2}\ dxdy.$

7. What is the region R over which you integrate when evaluating the iterated integral
$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}} \ \mathrm{d}y\ \mathrm{d}x?$$
Rewrite this as an iterated integral first with respect to x, then with respect to y. Evaluate this integral. Which order of integration is easier?

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define
$$u(x,t) = f(x - ct) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2,$$
where c is a real constant.
Verify that $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

9. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define
$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2+y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$
   (a) Define the vector field $F(x,y) = \nabla u(x,y)$. Express F in terms of f' and r.
   (b) Recall that the divergence of a vector field $F = P\,\widehat{i} + Q\,\widehat{j}$ is the scalar field given by $\text{div}F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Express the divergence of the gradient of u, in terms of f', f'' and r.
   The expression $\text{div}(\nabla u)$ is called the Laplacian of u, and is denoted by Δu or $\nabla^2 u$.

10. Let $f(x,y) = 4x - 7y$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 2x^2 + y^2$.
   (a) Sketch the graph of the set $C = g^{-1}(1) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1\}$.
   (b) Show that at the points where f has an extremum on C, the gradient of f is parallel to the gradient of g.
   (c) Find largest and the smallest value of f on C.

11. Let ω be the differential 1–form in $\mathbb{R}^3$ given by $\omega = x\,dx + y\,dy + z\,dz$.
   (a) Compute the differential, $d\omega$, of ω.
   (b) If possible, find a differential 0–form, f, such that $\omega = df$.
   (c) Let C be parametrized by a C^1 connecting $P_o(1,-1,-2)$ to $P_1(-1,1,2)$. Compute the line integral $\displaystyle\int_C \omega$.
   (d) Let C denote any simple closed curve in $\mathbb{R}^3$. Evaluate the line integral $\displaystyle\int_C \omega$.

12. Let f denote a differential 0–form in $\mathbb{R}^3$ and ω a a differential 1–form in $\mathbb{R}^3$.

    (a) Verify that $d(df) = 0$.

    (b) Verify that $d(d\omega) = 0$.

13. Let f and g denote differential 0–forms in $\mathbb{R}^3$, and ω and η a differential 1–forms in $\mathbb{R}^3$. Derive the following identities

    (a) $d(fg) = g\ df + f\ dg$.

    (b) $d(f\omega) = df \wedge \omega + f\ d\omega$.

    (c) $d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta$.

14. Let R denote the square, $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$, and ∂R denote the boundary of R oriented in the counterclockwise sense. Evaluate the line integral
$$\int_{\partial R} (y^2 + x^3)\ dx + x^4\ dy.$$