Assignment #6

Due on Wednesday, March 30, 2011

Read Section I.7 on *Autonomous Systems–Generalities,* pp. 37–46, in Hale's text.

Read Section I.8 on *Autonomous Systems–Limit Sets, Invariant Sets,* pp. 46–49, in Hale's text.

Read Chapter 4 on *Continuous Dynamical Systems,* starting on page 47, in the class lecture notes.

Background and Definitions

Let U denote an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. Let J_p denote the maximal interval of existence for the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p. \end{cases} \tag{1}$$

- *(Periodic Solutions)* A solution $u\colon J \to U$ of the differential equation in (1), which is not an equilibrium solution, is said to be periodic if there exists a positive number, τ, such that

  $$u(t+\tau) = u(t), \quad \text{for all } t \in J \text{ with } t+\tau \in J. \tag{2}$$

  The smallest positive number, τ, for which (2) holds true is called the period of u.

- *(Cycles)* In this problem set we will look at a condition that will guarantee that the solution to the IVP in (1) is periodic. We will also see that periodic solutions must be defined for all $t \in \mathbb{R}$. If the IVP in (1) has a periodic solution of period T, $u_p\colon \mathbb{R} \to U$, then the orbit of p, γ_p, is a closed curve parametrized by

  $$u_p\colon [0,T] \to U.$$

  The closed orbit, γ_p, is called a cycle.

Do the following problems

In problems 1 through 4, $u_p\colon J_p \to U$ denotes the unique solution to the IVP in (1).

1. Assume that there exist t_1 and t_2 in J_p such that $t_1 \neq t_2$ and
$$u_p(t_1) = u_p(t_2).$$
Prove that there exists $\tau > 0$ such that
$$u_p(t) = u_p(t+\tau), \qquad \text{for all } t \in J_p. \tag{3}$$

2. Prove that (3) implies that $J_p = \mathbb{R}$; that is, $u_p(t)$ is defined for all $t \in \mathbb{R}$.

   Suggestion: Write $J_p = (a, b)$ and assume, by way of contradiction, that $b \in \mathbb{R}$. Let (t_m) be a sequence in (a, b) such that t_m increases to b as $m \to \infty$ and $t_m - \tau \in J_p$ for all $m \in \mathbb{N}$. Show that $\lim\limits_{m\to\infty} u_p(t_m)$ exists in U.

3. Assume that p is not an equilibrium point of the system in (1) and define
$$T = \inf\{\tau > 0 \mid \text{ (3) holds true}\}. \tag{4}$$
Prove that $T > 0$ and $u_p(t) = u_p(t+T)$ for all $t \in \mathbb{R}$.

   Suggestion: Argue by contradiction; that is, assume that there exists a sequence, (τ_m), of positive numbers such that τ_m decreases to 0 and
$$u_p(t+\tau_m) = u_p(t), \qquad \text{for all } t \in \mathbb{R}.$$
Consider $\dfrac{u_p(t+\tau_m) - u_p(t)}{\tau_m}$ as $m \to \infty$.

4. Assume that p is not an equilibrium point of the system in (1), and that $u_p\colon \mathbb{R} \to U$ is a periodic solution on the IVP in (1). Show that the orbit, γ_p, is a cycle.

   Suggestion: Let T denote the period of the u_p. Show that
$$u_p\colon [0, T] \to U$$
is a parametrization of γ_p; that is,

   - $u_p\colon [0, T) \to U$ is one–to–one, and
   - $u_p([0, T]) = \gamma_p$.

5. Let $\theta\colon \mathbb{R} \times U \to U$ be a dynamical system in U. For $p \in U$, assume that γ_p is a cycle. Prove that $\omega(\gamma_p) = \gamma_p$, and $\alpha(\gamma_p) = \gamma_p$.

   Suggestion: Show that $\gamma_p \subseteq \omega(\gamma_p)$ and $\omega(\gamma_p) \subseteq \gamma_p$.