Assignment #7

Due on Wednesday, April 6, 2011

Read Section I.7 on *Autonomous Systems–Generalities,* pp. 37–46, in Hale's text.

Read Section I.8 on *Autonomous Systems–Limit Sets, Invariant Sets,* pp. 46–49, in Hale's text.

Read Chapter 4 on *Continuous Dynamical Systems,* starting on page 47, in the class lecture notes.

1. Assume that p is not an equilibrium point of the C^1 field, $F\colon U \to \mathbb{R}^N$, where U is an open subset of $\mathbb{R}^N$. Prove that if $\gamma_p^+ \cap \gamma_p^- \neq \emptyset$, then γ_p is a cycle.

2. Let U be an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. Let $u\colon \mathbb{R} \to U$ be a solution to the differential equation

$$\frac{dx}{dt} = F(x).$$

Suppose that there exists $q \in U$ such that

$$\lim_{t\to\infty} u(t) = q.$$

Prove that q must be an equilibrium point of F.

Suggestion: Write $F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$, where $f_j\colon U \to \mathbb{R}$, for $j = 1, 2, \ldots, N$, are C^1 functions. Arguing by contradiction, assume that, for some $j \in \{1, 2, \ldots, N\}$, $f_j(q) \neq 0$. Note that

$$u_j(t) = u_j(0) + \int_0^t f_j(u(\tau))\ d\tau, \qquad \text{for all } t \in \mathbb{R}.$$

You will need to show that, if $f_j(q) \neq 0$, there exists $\delta > 0$ such that

$$\|x - q\| < \delta_1 \Rightarrow |f_j(x)| > \frac{|f_j(q)|}{2}.$$

3. Consider the system

$$\begin{cases} \dfrac{dx}{dt} & = & y + \mu x^3; \\ \\ \dfrac{dy}{dt} & = & -x + \mu y^3, \end{cases} \tag{1}$$

where μ is a real parameter. For $(p,q) \in \mathbb{R}^2$, let $u_{(p,q)} \colon J_{(p,q)} \to \mathbb{R}^2$ denote the unique solution to the system in (1) subject to the initial condition

$$(x(0), y(0)) = (p, q), \tag{2}$$

where $J_{(p,q)}$ is the maximal interval of existence.

(a) Show that $(0,0)$ is the only equilibrium point of the system in (1).

(b) Assume that $\mu < 0$. Prove that if $\|(p,q)\| < \delta$, for some $\delta > 0$, then

$$\|u_{(p,q)}(t)\| < \delta, \text{ for all } t \in J_{(p,q)} \text{ with } t > 0.$$

Suggestion: Let $V(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$, and put

$$v(t) = V(u_{(p,q)}(t)), \quad \text{ for all } t \in J_{(p,q)}.$$

Show that if $(p,q) \neq (0,0)$, then $v(t)$ decreases as t increases. In other words, V decreases along the orbit $\gamma_{(p,q)}$.

(c) Assume that $\mu < 0$. Deduce from Part (b) that, if $\|(p,q)\| < \delta$, then $u_{(p,q)}(t)$ is defined for all $t \geqslant 0$.

4. *(Problem 3, Continued)* Assume that $\mu < 0$ in the system in (1). Prove that, for any $\varepsilon > 0$, if $\|(p,q)\| < \varepsilon$, then $\omega(\gamma_{(p,q)}) = \{(0,0)\}$.

Suggestion: Argue by contradiction following the following outline:

(i) Assume there exists $\varepsilon_o > 0$ and $(p_o, q_o) \neq (0,0)$ such that $\|(p_o, q_o)\| < \varepsilon_o$ and $\omega(\gamma_{(p_o,q_o)}) \neq \{(0,0)\}$. Explain why $\omega(\gamma_{(p_o,q_o)}) \neq \emptyset$. Thus, there exists $(\overline{x}, \overline{y}) \in \omega(\gamma_{(p_o,q_o)})$ with $(\overline{x}, \overline{y}) \neq (0,0)$.

(ii) Put $\theta(t, p_o, q_o) = u_{(p_o,q_o)}(t)$ for all $t \geqslant 0$. Explain why there exists a sequence of positive numbers, (t_m), such that $t_m \to \infty$ as $m \to \infty$ and

$$\lim_{m\to\infty} \theta(t_m, p_o, q_o) = (\overline{x}, \overline{y}).$$

(iii) Let $V(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$, and show that

$$V(\theta(t,p_o,q_o)) \geqslant V(\overline{x},\overline{y}), \qquad \text{for all } t > 0.$$

(iv) Show that

$$V(\theta(t,\overline{x},\overline{y})) < V(\overline{x},\overline{y}), \qquad \text{for all } t > 0.$$

(v) Show that there exists a δ_1, such that

$$\|(p,q) - (\overline{x},\overline{y})\| < \delta_1 \Rightarrow V(\theta(t,p,q) < V(\overline{x},\overline{y}), \qquad \text{for all } t > 0$$

(vi) Explain why there exists $M \in \mathbb{N}$ such that

$$m \geqslant M_1 \Rightarrow \|\theta(t_m,p_o,q_o) - (\overline{x},\overline{y})\| < \delta_1.$$

(vii) Put $(p,q) = \theta(t_{M_1},p_o,q_o)$, where M_1 is as given in the previous part. Explain why

$$\theta(t,p,q) = \theta(t + t_{M_1},p_o,q_o), \qquad \text{for all } t > 0,$$

and use this fact to derive a contradiction.

5. Let U be an open subset of $\mathbb{R}^N$ and let $V \colon U \to \mathbb{R}$ be a C^2 function. Put $F(x) = -\nabla V(x)$ for all $x \in U$. Assume that V has a (strict) local minimum at $\overline{x} \in U$; that is, there exists $r > 0$ such that $\overline{B_r(\overline{x})} \subset U$ and

$$V(\overline{x}) < V(y), \qquad \text{for all } y \in \overline{B_r(\overline{x})}\backslash\{\overline{x}\}.$$

Assume also that $\overline{B_r(\overline{x})}\backslash\{\overline{x}\}$ contains no equilibrium points of F.

(a) Show that $\overline{x}$ is an equilibrium point of the differential equation

$$\frac{dx}{dt} = F(x). \tag{3}$$

(b) Prove that there exists $\delta > 0$ such that, if $p \in B_\delta(\overline{x})$, the equation in (3) has a solution, $u_p \colon J_p \to U$, satisfying $u_p(0) = p$ and

$$u_p(t) \in \overline{B_r(\overline{x})}, \qquad \text{for all } t \in J_p \cap [0,t).$$

(c) Let $\delta > 0$ be as obtained in part (b). Deduce from the previous part that, if $p \in B_\delta(\overline{x})$, $u_p(t)$ is defined for all $t > 0$.

(d) Let $\delta > 0$ be as obtained in part (b). Prove that, if $p \in B_\delta(\overline{x})$, then $\omega(\gamma_p) = \{\overline{x}\}$.