Exam 1 (Part II)

Due on Monday, March 7, 2011

Name: ________________________________

This is the out–of–class portion of Exam 1. There is no time limit for working on the following two problems. You are allowed to consult your notes, the online class notes and the text for the course
Provide complete arguments when asked to prove a statement in a question. You will be graded on how well you organize your proofs as well as the logical flow or your deductions.
Write your name on this page and staple it to your solutions.

1. Let $A(t) = [a_{ij}]$ denote an $N \times N$ matrix whose entries are continuous functions, $a_{ij} \colon J \to \mathbb{R}$, defined over some open interval J. Consider the matrix differential equation
$$\frac{dY}{dt} = A(t)Y, \tag{1}$$
where $Y = Y(t)$ denotes a function whose values are $N \times N$ matrices.

   (a) Explain how you would apply the fundamental theory developed in Chapter 2 of the lecture notes to the system in (1) and deduce that the IVP,
$$\begin{cases} \dfrac{dY}{dt} = A(t)Y; \\ Y(t_o) = I, \end{cases} \tag{2}$$
where $t_o \in J$ and I denotes the $N \times N$ identity matrix, has a unique solution, $Y = Y(t)$, defined for all $t \in J$.

   (b) For $p \in \mathbb{R}^N$, define $u(t) = Y(t)p$, where $Y = Y(t)$ is the unique solution to the IVP in (2). Prove that $u = u(t)$ is the unique solution to the linear IVP
$$\begin{cases} \dfrac{dx}{dt} = A(t)x; \\ x(t_o) = p. \end{cases} \tag{3}$$

   (c) Verify that the matrix valued function
$$Y(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{t} - e^{-t} & e^{t} \end{pmatrix} \quad \text{for } t \in \mathbb{R},$$

satisfies the initial value problem in (1) where

$$A(t) = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}.$$

Use this fact and the result of part (b) to compute the flow map, $\theta(t,p,q)$, of the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= -x; \\ \\ \dfrac{dy}{dt} &= 2x + y. \end{cases}$$

2. Consider the following initial value problem for a second order equation:

$$\begin{cases} \dfrac{d^2u}{dt^2} + g(u) = 0; \\ \\ u(0) = p \\ \\ u'(0) = q, \end{cases} \tag{4}$$

where p and q are given real values, and $g\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function.

(a) Prove that the IVP in (4) has a unique solution, $u = u(t,p,q)$, defined on a maximal interval of existence, $J_{(p,q)}$, for each $(p,q) \in \mathbb{R}^2$.

(b) Prove that $u(t,p,q)$, defined in part (a), depends continuously on (p,q).

(c) Give a condition on g that will guarantee that the solution to the IVP in (4) exists for all $t \in \mathbb{R}$. Prove your assertion.