Exam 2

Wednesday, April 27, 2011

Name: ______________________________

This is a closed–notes and closed–book exam. Use your own paper and/or the paper provided for you. Please, provide complete answers. Write you name on this page and staple it to your work. You have 75 minutes to work on the following 4 problems. Relax.

Background and Notation
All the questions in this exam refer to the system

$$\frac{dx}{dt} = F(x), \tag{1}$$

where $F\colon U \to \mathbb{R}^N$ is a C^1 vector field defined on an open subset, U, of $\mathbb{R}^N$.

The function $u_p\colon J_p \to \mathbb{R}^N$ denotes the unique solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} &= F(x); \\ x(0) &= p, \end{cases} \tag{2}$$

defined on the maximal interval of existence, J_p.

Answer the Following Questions

1. Let $V\colon U \to \mathbb{R}$ denote a C^1 function.

   (a) Define the Lie derivative, $\dot{V}\colon U \to \mathbb{R}$, of V along the flow of F and explain its significance.

   (b) State what it means for V to be a Liapunov function for the system in (1).

   (c) Let $\overline{x} \in U$ denote an equilibrium point of F. Give precise definitions for the following statements:
   (i) $\overline{x}$ is isolated; (ii) $\overline{x}$ is stable; (iii) $\overline{x}$ is asymptotically stable; (iv) $\overline{x}$ is unstable.

   (d) Without proof, give conditions on V and $\dot{V}$ that will guarantee that and isolated equilibrium point, $\overline{x}$, of F is
   (i) stable; (ii) asymptotically stable; (iii) unstable.

2. Let p denote any point in U.

   (a) Define the orbit, γ_p, of p under the flow of F.

   (b) Give a precise definition of what it means for a subset of U to be invariant under the flow of F, and prove that γ_p is invariant.

   (c) Define the ω–limit set, $\omega(\gamma_p)$, of γ_p and and give, without proof, a condition that will guarantee that $\omega(\gamma_p)$ is non–empty.

   (d) Under the condition given in the previous part, give three properties of $\omega(\gamma_p)$, in addition to it being non–empty.

3. Let γ denote any orbit of the system in (1).

   (a) State precisely what it means for γ to be a cycle.

   (b) Without proof, state conditions that will guarantee that an orbit γ_p of the system in (1) is a cycle.

   (c) Assume that γ is a cycle of the system in (1). Prove that $\omega(\gamma) = \gamma$.

   (d) State what it means for a cycle, γ, to be isolated.

   (e) State precisely what it means for an isolated cycle, γ, to be a limit cycle.

4. Let $V\colon U \to \mathbb{R}$ be a Liapunov function for the system in (1) over the open set U. Let $p \in U$ and denote by $u_p\colon J_p \to U$ the unique solution to the IVP in (2) defined on the maximal interval of exitence J_p. Suppose also that the set

$$\{u_p(t) \mid t \in J_p \cap [0, \infty)\}$$

   is bounded.

   (a) Prove that $u_p(t)$ is defined for all $t \geqslant 0$.

   (b) Assume, in addition, that there exists a real constant, c, such that

$$\lim_{t\to\infty} V(u_p(t)) = c.$$

   Prove that

$$V(\overline{y}) = c, \qquad \text{for all } \overline{y} \in \omega(\gamma_p).$$

   Deduce then that

$$\dot{V}(\overline{y}) = 0, \qquad \text{for all } \overline{y} \in \omega(\gamma_p).$$