Notes on Dynamical Systems

Preliminary Lecture Notes

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Chapter 1

Introduction

These notes provide an introduction to the theory of dynamical systems. We will begin by proving the fundamental existence and uniqueness theorem for initial value problem for a system of first–order, ordinary differential equations. We will then proceed to establish results dealing with continuous dependence on initial conditions and parameters, and the question of extending solutions. The main goal on this part of the course will be the definition of a flow of a vector field in $\mathbb{R}^N$. We will then study properties of flows, which are also known as dynamical systems.

1.1 Integral Curves

We will presently formulate one of the fundamental question in the study of dynamical systems that we will be studying in this course.

Let U be an open subset in $\mathbb{R}^N$ and I be an open interval in $\mathbb{R}$. Consider a vector valued function, $F\colon I \times U \to \mathbb{R}^N$, defined on $I \times U$, given by

$$F(t, x_1, x_2, \ldots, x_N) = \begin{pmatrix} f_1(t, x_1, x_2, \ldots, x_N) \\ f_2(t, x_1, x_2, \ldots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \ldots, x_N) \end{pmatrix}, \tag{1.1}$$

where $f_1, f_2, \ldots, f_N$ are real valued functions defined on $I \times U$. F is called a (time dependent) vector field defined on U; at each point $(x_1, x_2, \ldots, x_N)$ in the open set U, and at "time" $t \in I$, $F(t, x_1, x_2, \ldots, x_N)$ gives a vector in $\mathbb{R}^N$.

Let J be an open interval contained in I. A vector valued function

$$u\colon J \to U$$

given by

$$u(t) = (x_1(t), x_2(t), \ldots, x_N(t)),$$

is said to be a C^1 curve in U if all the component functions

$$x_i \colon J \to \mathbb{R}, \qquad \text{for } i = 1, 2, \ldots, N,$$

are differentiable with continuous derivatives, $x_i'(t)$, for $i = 1, 2, \ldots, N$. The vector

$$u'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_N'(t) \end{pmatrix}$$

is called the tangent vector to the curve u at $u(t)$, for $t \in J$.

The first question that we would like to answer is the the following: Given a point $p_o \in U$ and a time $t_o \in I$, is there a C^1 curve, $u \colon J \to U$, defined on some subinterval, J, of I which contains t_o, such that

$$u(t_o) = p_o,$$

and

$$u'(t) = F(t, u(t)), \qquad \text{for all } t \in J?$$

In other words, given a point, p_o, in U, is it possible to find a C^1 curve through p_o when $t = t_o$, and such that its tangent vectors are prescribed by the vector field F? If this is the case, we say that $x(t) = u(t)$, for $t \in J$, solves the initial value problem (IVP)

$$\begin{cases} \dfrac{dx}{dt} = F(t, x); \\ \\ x(t_o) = p_o. \end{cases} \tag{1.2}$$

Here, $x = x(t)$ denotes the vector valued function

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix},$$

and $\dfrac{dx}{dt}$ is the tangent vector at t,

$$\frac{dx}{dt} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_N'(t) \end{pmatrix}.$$

If a C^1 curve, $u \colon J \to U$, satisfying the IVP (1.2) exists, we will call it an integral curve of the vector field F going through the point p_o when $t = t_o$. We will see in subsequent chapters that for an integral curve to exist, it is sufficient that the vector field, F, be continuous in a neighborhood of the point (t_o, p_o).

Example 1.1.1. In this one–dimensional example, we let $U = \mathbb{R}$, $t_o = 0$, $p_o = 0$, and

$$F(t, x) = tx^{1/3}, \qquad \text{for all } (t, x) \in \mathbb{R}^2.$$

We find a solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t, x); \\ \\ x(0) = 0, \end{cases} \tag{1.3}$$

by first separating variables in the ODE

$$\frac{dx}{dt} = tx^{1/3}$$

to get

$$3x^{2/3} = t^2 + c, \tag{1.4}$$

for some arbitrary constant, c. Using the initial condition in the IVP (1.3), we obtain from (1.4) that $c = 0$, so that

$$x(t) = \frac{t^3}{3\sqrt{3}}, \qquad \text{for } t \in \mathbb{R}$$

solves the IVP (1.3).

Note that the function defined by

$$x(t) = 0, \qquad \text{for all } t \in \mathbb{R},$$

also solves the IVP (1.3). Thus, uniqueness of a solution to the IVP (1.2) is not, in general, guaranteed for a continuous vector field, F.

1.2 Flows

One of our goals in this course is to define the flow of a vector field, F. Heuristically, the flow of a vector field, $F: I \times U \to U$, where I is an open interval containing 0, is a map, θ, from a domain $\mathcal{D} \subseteq I \times U$ to U, which assigns to each point $(t, p) \in \mathcal{D}$, the value of $u_p(t)$, where $u_p: J_p \to U$ is an integral curve of F which solves the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t, x); \\ \\ x(0) = p. \end{cases} \tag{1.5}$$

(The interval of existence, J_p, for u_p is assumed here to contain both 0 and t). Thus,

$$\theta(t, p) = u_p(t), \qquad \text{for } (t, p) \in \mathcal{D}.$$

Observe that for the map, θ, to be well defined, through each point $p \in U$ there must exist a most one integral curve, u_p, of F. As was pointed out in Example

1.1.1, the IVP (1.2) is not always guaranteed to have a unique solution. We will see in the next chapter that uniqueness of the integral curve going through a point (t_o, p_o) in $I \times U$ is guaranteed for the case in which, in addition to the vector field, F, being continuous, the field $F(t,x)$ is assumed to satisfy a local Lipschitz condition in the second variable; that is, there exists a ball of radius $r_o > 0$ around p_o, denoted by $B_{r_o}(p_o)$, such that $\overline{B}_{r_o}(p_o) \subset U$ (here, $\overline{B}_{r_o}(p_o)$ denotes the closure of the open ball $B_{r_o}(p_o) = \{x \in \mathbb{R}^n \mid \|x - p_o\| < r_o\}$; that is, $\overline{B}_{r_o}(p_o) = \{x \in \mathbb{R}^n \mid \|x - p_o\| \leqslant r_o\}$); a positive number $\delta_o > 0$ such that $[t_o - \delta_o, t_o + \delta_o] \subset I$; and a constant K_o, depending on p_o, for which

$$\|F(t,x) - F(t,y)\| \leqslant K_o \|x - y\|, \quad \text{for } x, y \in \overline{B}_{r_o}(p_o), \text{ and } \ |t - t_o| \leqslant \delta_o.$$

The symbol $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^N$; in other words,

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}.$$

An example of a vector field for which a local Lipschitz condition holds is a C^1 vector field, F; that is, a vector field as given in (1.1) in which the component functions have continuous partial derivatives

$$\frac{\partial f_i}{\partial t}, \ \frac{\partial f_i}{\partial x_j}, \quad \text{for } i, j = 1, 2, \ldots, N.$$

We will also need to give a more precise definition of the domain, $\mathcal{D}$, for the flow map, θ. This will entail knowing that each integral curve, u_p, through $p \in U$ is defined on a maximal interval of existence J_p containing t_o; that is, J_p contains any open interval containing t_o on which a integral curve satisfying the IVP (1.5) is defined. This will also be proved in the next chapter.

Finally, we also want the flow map,

$$\theta \colon \mathcal{D} \to U$$

to be at least continuous. This will require proving results regarding continuous dependence on initial conditions. We will prove these continuity results for the case in which F is Lipschitz continuous in the next chapter.

Chapter 2

Fundamental Existence Theory for Ordinary Differential Equations

Let I denote an open interval and U be an open subset of $\mathbb{R}^N$. Assume that

$$F\colon I\times U\to U$$

is a continuous vector field, which satisfies a local Lipschitz condition at $(t_o, p_o)\in I\times U$; that is, there exist positive numbers r_o and δ_o such that $\overline{B}_{r_o}(p_o)\subset U$ and $[t_o-\delta_o, t_o+\delta_o]\subset I$, and a constant K_o for which

$$\|F(t,x)-F(t,y)\|\leqslant K_o\|x-y\|, \quad \text{for } x,y\in\overline{B}_{r_o}(p_o), \text{ and } \ |t-t_o|\leqslant\delta_o. \tag{2.1}$$

We begin this chapter by proving that there there exists a positive constant, δ, such that $\delta<\delta_o$ and a C^1 curve

$$u\colon (t_o-\delta, t_o+\delta)\to U$$

such that

$$\gamma(t_o)=p_o,$$

and

$$u'(t)=F(t,\gamma(t)), \qquad \text{for all } t\in(t_o-\delta, t_o+\delta).$$

We shall refer to this as the Local Existence and Uniqueness Theorem for Lipschitz continuous vector fields. We will rephrase it here for future reference.

Theorem 2.0.1 (Local Existence and Uniqueness Theorem)**.** Let $I\subseteq\mathbb{R}$ be an open interval, and $U\subseteq\mathbb{R}^N$ be an open set. Let $(t_o,p_o)\in I\times U$ and suppose that the vector field $F\colon I\times U\to U$ satisfies a local Lipschitz condition at (t_o,p_o) given in (2.1). Then, there exists $\delta>0$ and a C^1 function

$$u\colon (t_o-\delta, t_o+\delta)\to U$$

which solves the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t, x); \\ \\ x(t_o) = p_o. \end{cases} \tag{2.2}$$

over the interval $(t_o - \delta, t_o + \delta)$.

The main goal of this chapter is to provide a proof of Theorem 2.0.1 and its extensions. In the next section we set up the framework that will be needed in the proof.

2.1 Setup for the Proof of Theorem 2.0.1

We first look for a local solution of IVP (2.2) among a class of functions defined on some bounded interval $[a, b]$ contained in $(t_o - \delta_o, t_o + \delta_o)$ and which contains t_o. We begin by considering the class of continuous functions,

$$u\colon [a, b] \to \mathbb{R}^N,$$

defined on $[a, b]$. We will denote the class by $C[a, b]$. This is a vector space which can be endowed with the norm

$$\|u\| = \max_{t\in[a,b]} \|u(t)\|, \quad \text{for all } u \in C[a, b]. \tag{2.3}$$

Observe that the symbol $\|\cdot\|$ in (2.3) is used to mean two things: (1) to the right of the equal sign in (2.3), $\|u(t)\|$ denotes the Euclidean norm of the vector $u(t)$ in $\mathbb{R}^N$; (2) to the left of the equal sign in (2.3), $\|u\|$ denotes the norm of the function $u\colon [a, b] \to \mathbb{R}^N$ being defined by the expression in (2.3). The context will make clear which sense of the symbol $\|\cdot\|$ is being used.

The norm, $\|\cdot\|$, defined in (2.3) gives rise to metric, or distance function, $d(\cdot, \cdot)$, in the space $C[a, b]$:

$$d(u, v) = \|u - v\|, \quad \text{for all } u, v \in C[a, b]. \tag{2.4}$$

The space $C[a, b]$ together with the metric, $d(\cdot, \cdot)$ is a complete metric space; meaning that any Cauchy sequence of functions, (u_m), in $C[a, b]$, converges to a function in $C[a, b]$.

Definition 2.1.1 (Cauchy Sequence in $C(a, b)$)**.** A sequence of functions, (u_m), in $C[a, b]$ is said to be a Cauchy sequence if, for any $\varepsilon > 0$, there exists $M \in \mathbf{N}$ such that

$$m, n \geqslant M \Rightarrow d(u_m, u_n) < \varepsilon,$$

where $d(\cdot, \cdot)$ is the metric in $C[a, b]$ defined in (2.4).

Proposition 2.1.2 (Completeness of $C(a, b)$)**.** Every Cauchy sequence of functions, (u_m), in $C[a, b]$ converges to a continuous function, $u\colon [a, b] \to \mathbb{R}^N$, in the sense that

$$\lim_{m\to\infty} \|u_m - u\| = 0,$$

where $\|\cdot\|$ is the norm in $C[a, b]$ defined by the expression in (2.3).

A proof of Proposition 2.1.2 may be found in various texts on Real Analysis. For instance, in [Bar76], it is stated as the Cauchy Criterion for Uniform Convergence on page 121.

Definition 2.1.3 (Banach Space)**.** A vector space with norm, $\|\cdot\|$, which is complete with respect to the metric generated by that norm is called a Banach space.

Thus, Proposition 2.1.2 states that the space $C[a,b]$ with the norm, $\|\cdot\|$, defined in (2.3) is a Banach space.

Let I denote an open interval and U be an open subset in $\mathbb{R}^N$. Suppose that the vector field

$$F\colon I\times U\to\mathbb{R}^N$$

is continuous. Let $(t_o,p_o)\in I\times U$ and let $[a,b]$ be a closed interval containing t_o in its interior with $[a,b]\in I$. Given a function $u\in C[a,b]$ with the property that $u(t)\in U$ for all $t\in[a,b]$, we can define a new function

$$T(u)\colon [a,b]\to\mathbb{R}^N$$

as follows:

$$T(u)(t)=p_o+\int_{t_o}^{t}F(\tau,u(\tau))\ \mathrm{d}\tau,\qquad \text{for all } t\in[a,b], \tag{2.5}$$

where the integral on the right–hand side of the equal sign in (2.5) is to be understood as a vector in $\mathbb{R}^N$ with components given by

$$\int_{t_o}^{t}f_i(\tau,u(\tau))\ \mathrm{d}\tau,\qquad \text{for } i=1,2,\ldots,N.$$

Put

$$v(t)=T(u)(t),\qquad \text{for all } t\in[a,b], \tag{2.6}$$

where $T(u)\colon [a,b]\to\mathbb{R}^N$ is as defined in (2.5). It then follows from (2.6), (2.5) and the Fundamental Theorem of Calculus that v is differentiable over (a,b), with

$$v'(t)=F(t,u(t)),\qquad \text{for all } t\in(a,b). \tag{2.7}$$

Furthermore,

$$v(t_o)=p_o. \tag{2.8}$$

The expressions in (2.7) and (2.8) suggest one way to prove the existence of a solution to the IVP in (2.2) defined over some interval (a,b) containing t_o. Suppose that we can find a solution, u, to the fixed–point equation

$$T(u)(t)=u(t),\qquad \text{for } t\in[a,b]. \tag{2.9}$$

Then, that solution will satisfy

$$u(t_o)=p_o \tag{2.10}$$

and

$$u'(t) = F(t, u(t)), \qquad \text{for all } t \in (a, b), \tag{2.11}$$

by virtue of (2.7) and (2.8). In other words, u solves the IVP (2.2) over the interval (a, b).

Observe from the fixed–point equation in (2.9) that any solution of (2.9) must lie in the image of T as well as the domain of T. The domain of T consists of functions u in $C[a, b]$ such that $u(t) \in U$ for all $t \in [a, b]$. From the definition of the map T in (2.5) we see that it is not necessarily the case that, if $u(t) \in U$ for all $t \in [a, b]$, then $T(u)(t) \in U$ for all $t \in [a, b]$. Thus, in order to find a solution of the fixed–point equation in (2.9) we will need to restrict the domain of T so that $T(u)(t) \in U$ for all $t \in [a, b]$ for all functions, u, in the restricted domain. We can do this by using the assumption that U is open. Let $B \equiv B_r(p_o)$, where $0 < r < r_o$, be an open ball around p_o such that $\overline{B} \subset U$. We can then define a subset of $C[a, b]$, which we will denote by $C([a, b], \overline{B})$, as follows:

$$C([a, b], \overline{B}) = \{u \in C[a, b] \mid u(t) \in \overline{B} \ \text{ for all } t \in [a, b]\}.$$

Thus, the map T is defined on $C([a, b], \overline{B})$ and it remains to show that the interval $[a, b]$ can be chosen so that T maps $C([a, b], \overline{B})$ to itself. We can do this by taking advantage of the assumption that the vector field, F, is continuous so that it is bounded on the compact set $[t_o - \delta_o, t_o + \delta_o] \times \overline{B}_{r_o}(p_o)$, where δ_o and r_o are as given in the statement of the Lipschitz condition in (2.1). In fact, let

$$M_o = \max_{\substack{|t-t_o|\leqslant\delta_o \\ x\in\overline{B}_{r_o}(p_o)}} \|F(t, x)\|; \tag{2.12}$$

then, for any $u \in C([a, b], \overline{B})$, using (2.5), we obtain that

$$\begin{aligned} \|T(u)(t) - p_o\| &\leqslant \left| \int_{t_o}^{t} \|F(\tau, u(\tau))\| \, \mathrm{d}\tau \right| \\ &\leqslant M_o |t - t_o|, \end{aligned} \tag{2.13}$$

for all $t \in [a, b]$, where we have also used (2.12). Thus, if we choose $[a, b]$ *a priori* to be $[t_o - \delta, t_o + \delta]$, for some $\delta > 0$ with $\delta < \delta_o$, we obtain from (2.13) that

$$\|T(u)(t) - p_o\| \ \leqslant \ M_o \delta, \qquad \text{for all } t \in [a, b]. \tag{2.14}$$

Thus, by choosing δ so that

$$\delta \leqslant \frac{r}{M_o}, \tag{2.15}$$

we get from (2.14) and (2.15) that

$$\|T(u)(t) - p_o\| \ \leqslant \ r, \qquad \text{for all } t \in [a, b], \tag{2.16}$$

where

$$[a, b] = [t_o - \delta, t_o + \delta]. \tag{2.17}$$

It then follows from (2.16) that, if $[a, b]$ is given as in (2.17), where δ is chosen so that $\delta < \delta_o$ and (2.15) holds, then

$$T\colon C([a, b], \overline{B}) \to C([a, b], \overline{B}); \tag{2.18}$$

in other words, T maps the metric space $C([a, b], \overline{B})$ to itself.

In order to prove that the map T defined in (2.5) has a fixed point, we may apply one of various fixed point theorems to appropriately chosen $[a, b]$ and B. We will apply the Contraction Mapping Principle of Banach.

Definition 2.1.4 (Contraction)**.** Let X denote a metric space with metric $d(\cdot, \cdot)$. A function $T\colon X \to X$ is said to be a contraction if there exists $k > 0$ such that $k < 1$ and

$$d(T(x), T(y)) \leqslant k\ d(x, y), \quad \text{for all } x, y \in X. \tag{2.19}$$

Remark 2.1.5. If the metric in X is generated by a norm, $\|\cdot\|$, then (2.19) in the definition of a contraction reads

$$\|T(x) - T(y)\| \leqslant k\ \|x - y\|, \quad \text{for all } x, y \in X. \tag{2.20}$$

Theorem 2.1.6 (Contraction Mapping Principle)**.** Let X be a complete metric space and $T\colon X \to X$ be a contraction. Then, T has a unique fixed point in X. That is, there exists a unique $u \in X$ such that

$$T(u) = u.$$

Theorem 2.1.6 is proved in various Real Analysis books. In the text for this course, [Hal09], it is presented as the Contraction Mapping Principle of Banach-Cacciopoli [Hal09, Theorem 3.1] and proved on page 5.

The proof of the local existence and uniqueness theorem for locally Lipschitz continuous vector fields, which we will present in the next section, boils down to proving that there is an appropriately chosen interval, $[a, b]$, for which the map

$$T\colon C([a, b], \overline{B}) \to C([a, b], \overline{B}),$$

defined in (2.5), is a contraction. The proof of Theorem 2.0.1 will then follow from the Contraction Mapping Principle (Theorem 2.1.6).

2.2 Proof of the Local Existence and Uniqueness Theorem

In the previous section we saw that, if $[a, b] = [t_o - \delta, t_o + \delta]$, where $\delta > 0$ satisfies $\delta < \delta_o$ and (2.15), then the map T defined in (2.5) maps the metric space $C([a, b], \overline{B})$ to itself. We will now show that, by restricting δ further, T will be a contraction.

Let $u, v \in C([a,b], \overline{B})$ and consider

$$T(u)(t) - T(v)(t) = \int_{t_o}^{t} (F(\tau, u(\tau)) - F(\tau, v(\tau)))\, \mathrm{d}\tau, \quad \text{for all } t \in [a,b], \tag{2.21}$$

where we have used the definition of the map T in (2.5). Since $\overline{B} \subset B_{r_o}(p_o)$, we can apply the Lipschitz condition in (2.1) to obtain from (2.21) that

$$\|T(u)(t) - T(v)(t)\| \leqslant \left| \int_{t_o}^{t} K_o \|u(\tau) - v(\tau)\|\, \mathrm{d}\tau \right|, \quad \text{for all } t \in [a,b]. \tag{2.22}$$

Thus, using the definition of the norm in $C([a,b], \overline{B})$ in (2.3), we get from (2.22) that

$$\|T(u)(t) - T(v)(t)\| \leqslant K_o |t - t_o| \|u - v\|, \quad \text{for all } t \in [a,b],$$

which implies that

$$\|T(u) - T(v)\| \leqslant K_o\ \delta\ \|u - v\|, \tag{2.23}$$

by virtue of the definition of the norm in $C([a,b], \overline{B})$. Consequently, setting

$$k = K_o \delta, \tag{2.24}$$

we obtain from (2.23) that

$$\|T(u) - T(v)\| \leqslant k\ \|u - v\|. \tag{2.25}$$

Thus, if we restrict δ further so that

$$\delta < \frac{1}{K_o}, \tag{2.26}$$

we see from (2.24) and (2.25) that T is a contraction.

We are now in a position to prove the Fundamental Theorem of Ordinary Differential Equations.

Proof of Theorem 2.0.1: Choose $\delta > 0$ so that

$$\delta < \min \left\{ \delta_o, \frac{r}{M_o}, \frac{1}{K_o} \right\}, \tag{2.27}$$

where $r > 0$ is chosen so that $r < r_o$ and

$$B = B_r(p_o) \subset U;$$

the constants δ_o, r_o and K_o are given by the Lipschitz condition assumption on the vector field

$$F : I \times U \to U,$$

which was stated in (2.1); and M_o is given in (2.12); namely,

$$M_o = \max_{\substack{|t - t_o| \leqslant \delta_o \\ x \in \overline{B}_{r_o}(p_o)}} \|F(t,x)\|.$$

Put $[a,b] = [t_o - \delta, t_o + \delta]$. It then follows from (2.27), the calculations in the previous set-up section, (2.24) and (2.25) that the map

$$T\colon C([a,b],\overline{B}) \to C([a,b],\overline{B}),$$

defined in (2.5), is a contraction. Consequently, by the Contraction Mapping Principle (Theorem 2.1.6), there exists a unique function $u \in C([a,b],\overline{B})$ satisfying

$$u(t) = T(u)(t), \quad \text{for all } t \in [t_o - \delta, t_o + \delta],$$

or

$$u(t) = p_o + \int_{t_o}^{t} F(\tau, u(\tau))\, \mathrm{d}\tau, \quad \text{for all } t \in [t_o - \delta, t_o + \delta]. \tag{2.28}$$

It then follows from (2.28) and (2.9)–(2.11) in Section 2.1, that u is the unique solution to the IVP (2.2) over the interval $(t_o - \delta, t_o + \delta)$. This completes the proof of the Fundamental Theorem for Ordinary Differential Equations. ■

Remark 2.2.1 (Another Proof of Uniqueness)**.** Uniqueness of the solution of the IVP (2.2) over the interval of existence $[t_o - \delta, t_o + \delta]$, where

$$\delta K_o < 1, \tag{2.29}$$

for the case in which the vector field, F, satisfies a local Lipschitz condition as described in (2.1), follows from the uniqueness assertion of the Contraction Mapping Principle. However, it can also be proved directly as a consequence of the local Lipschitz condition and the integral representation (2.28) for a solution to the IVP (2.2). We present the independent proof of uniqueness in this remark because it uses calculations and results that will be needed in subsequent sections.

Put $\overline{J} = [t_o - \delta, t_o + \delta]$, where δ satisfies (2.29), and suppose that $u\colon \overline{J} \to U$ and $v\colon \overline{J} \to V$ both solve the differential equation (DE)

$$\frac{dx}{dt} = F(t,x) \tag{2.30}$$

over the open interval, J, contained in I. Furthermore, assume that J is small enough so that $\overline{J} \subset (t_o - \delta_o, t_o + \delta_o)$ and $u(t), v(t) \in B_{r_o}(p_o)$, for all $t \in \overline{J}$.

Put

$$w(t) = u(t) - v(t), \quad \text{for all } t \in \overline{J}. \tag{2.31}$$

Observe that if both v and u solve the IVP (2.2) over J, then

$$w(t_o) = 0, \tag{2.32}$$

by virtue of (2.31). Put

$$g(t) = \|w(t)\|, \quad \text{for all } t \in \overline{J}. \tag{2.33}$$

Then, g is continuous over $\overline{J}$ and it therefore takes on a maximum value, M, over the compact interval $\overline{J}$; that is,

$$M = \max_{t\in\overline{J}} g(t) \tag{2.34}$$

is attained at some point $t_1 \in \overline{J}$; so that

$$M = g(t_1). \tag{2.35}$$

Observe that

$$w(t_1) - w(t_o) = \int_{t_o}^{t_1} w'(\tau)\ \mathrm{d}\tau. \tag{2.36}$$

Thus, using (2.32), (2.31) and (2.30), we obtain from (2.36) that

$$w(t_1) = \int_{t_o}^{t_1} (F(\tau, u(\tau)) - F(\tau, v(\tau)))\ \mathrm{d}\tau. \tag{2.37}$$

Thus, applying the Lipschitz condition in (2.1) to the expression in (2.37), we get

$$\|w(t_1)\| \leqslant K_o \left| \int_{t_o}^{t_1} \|u(\tau) - v(\tau)\|\ \mathrm{d}\tau \right|,$$

from which we get

$$g(t_1) \leqslant K_o \left| \int_{t_o}^{t_1} g(\tau)\ \mathrm{d}\tau \right|, \tag{2.38}$$

by virtue of (2.31) and (2.33). Consequently, using (2.34) and (2.35), we obtain from (2.38),

$$M \leqslant K_o \delta M. \tag{2.39}$$

We claim that

$$M = 0. \tag{2.40}$$

Otherwise, we would obtain from (2.39) that

$$1 \leqslant K_o \delta,$$

which is in direct contradiction with (2.29). Thus, (2.40) must be true, and so, by virtue of (2.34), (2.33) and (2.31),

$$u(t) = v(t), \quad \text{for all } t \in \overline{J}.$$

2.3 Extension of Solutions

In the previous section, we obtained existence of a unique solution to the IVP (2.2) in some interval $(t_o - \delta, t_o + \delta)$ around t_o for the case in which the vector field, $F(t, x)$, satisfies a local Lipschiptz condition in the second variable as described in (2.1). In this section we prove that a unique solution can be obtained in a maximal interval of existence, J_{p_o}, where J_{p_o} is an open interval containing t_o such that $J_{p_o} \subseteq I$.

Example 2.3.1. Let $U = \mathbb{R}^2$ and let $F: U \to \mathbb{R}^2$ denote the vector field given by

$$F(x,y) = \begin{pmatrix} x^2 \\ 0 \end{pmatrix};$$

$t_o = 0$ and $p_o = (1,0)$. In this case the IVP (2.2) leads to the two–dimensional system

$$\begin{cases} \dfrac{dx}{dt} = x^2; \\ \\ \dfrac{dy}{dt} = 0, \end{cases} \tag{2.41}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 0. \tag{2.42}$$

Solving the system in (2.41), we are lead to the solution

$$u(t) = \left(\frac{1}{1-t}, \ 0 \right), \qquad \text{for } t < 1,$$

satisfying the initial condition in (2.42). Thus, in this case, the maximal interval of existence is $J_{(1,0)} = (-\infty, 1)$. Note that, as t approaches 1 from the left,

$$\|u(t)\| = \frac{1}{1-t} \to +\infty.$$

Remark 2.3.2. Observe that the vector field, F, in Example 2.3.1 does not depend explicitly on t. Thus, the system in (2.41) that it generates does not depend explicitly on t either. Systems that are explicitly independent of the "time" variable, t, are said to be autonomous. We will spend a large portion of this course studying autonomous systems.

Remark 2.3.3. Note that the domain, $I \times U$, of the vector field generating the autonomous system in Example 2.3.1 can be thought to be $\mathbb{R} \times U$; that is, $I = \mathbb{R}$ for autonomous systems. The result of Example 2.3.1 shows that even in the case of an autonomous systems, existence of a solution defined for all $t \in \mathbb{R}$ is not guaranteed. Later in this section, we will see which conditions on the vector field will imply existence of solutions which are defined for all times, t. Such solutions are known as global solutions.

In order to prove the main extension result of this section, we begin with the following lemma.

Lemma 2.3.4. Let $I \subseteq \mathbb{R}$ be an open interval, and $U \subseteq \mathbb{R}^N$ be an open set. Suppose that the vector field $F: I \times U \to U$ satisfies a local Lipschitz condition in the second variable at every $(t,p) \in I \times U$, and let u and v denote two C^1 solutions of the differential equation

$$\frac{dx}{dt} = F(t,x), \tag{2.43}$$

which are defined in an open interval $J \subseteq I$. If u and v agree at a point $t_o \in J$; that is,

$$u(t_o) = v(t_o), \tag{2.44}$$

then

$$u(t) = v(t), \quad \text{for all } t \in J. \tag{2.45}$$

Proof: Define a subset, J_o, of J as follows:

$$J_o = \{t \in J \mid u(t) = v(t)\}.$$

Then, $t_o \in J_o$, by (2.44); so that $J_o \neq \emptyset$. Furthermore, J_o is closed; since J_o is the pre-image under $u - v$ of the closed set $\{0\}$, where 0 denotes the zero vector in $\mathbb{R}^N$. We will show that J_o is also open. In fact, let t_1 be any point in J_o and put $p_1 = v(t_1) = v(t_1)$. Then, by the local existence and uniqueness theorem for ODEs, Theorem 2.0.1, the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t, x); \\ \\ x(t_1) = p_1, \end{cases}$$

has a unique solution defined in an open interval $J_1 = (t_1 - \delta, t_1 + \delta)$. We may assume that $\delta > 0$ is small enough so that $J_1 \subset J$. By uniqueness, we get that $u(t) = v(t)$ for all $t \in J_1$, since $u(t_1) = v(t_1)$. It then follows that $J_1 \subset J_o$. Hence, J_o is also open. Thus, J_o is a non–empty, closed subset of J which is also open. Hence, by the connectedness of J, $J_o = J$. Thus, u and v agree on all of J, which we wanted to show. ■

Given a point $(t_o, p_o) \in I \times U$, let $\mathcal{J}_{p_o}$ denote the collection of all open intervals, J, that contain t_o and on which there is defined a C^1 solution, u, of the differential equation in (2.43) with $u(t_o) = p_o$. Define

$$J_{p_o} = \bigcup \mathcal{J}_{p_o}; \tag{2.46}$$

that is, J_{p_o} is the union of all intervals in $\mathcal{J}_{p_o}$. Next, define a function,

$$u_{p_o} \colon J_{p_o} \to U,$$

by

$$u_{p_o}(t) = u(t), \quad \text{for } t \in J_{p_o}, \tag{2.47}$$

where u is a solution to the differential equation in (2.43) defined on $J \in \mathcal{J}_{p_o}$ with $t \in J$ and $u(t_o) = p_o$. By the result of Lemma 2.3.4, if $v \colon \widetilde{J} \to U$ is another solution to the differential equation in (2.43) defined in $\widetilde{J} \in \mathcal{J}_{p_o}$ with $t \in \widetilde{J}$ and $v(t_o) = p_o$, then $v(t) = u(t)$, since $v(t_o) = u(t_o)$, and so v and u agree on any open interval that contains t_o. Thus, the function, u_{p_o}, defined in (2.47) is well defined. Observe that u_{p_o} solves IVP (2.2) over the entire interval J_{p_o}.

The open interval J_{p_o} defined in (2.46) is called the maximal interval of existence for the IVP (2.2). The interval J_{p_o} is maximal in the sense that J_{p_o} cannot be a proper subset of an open interval on which there is defined a solution to the IVP (2.2). This is evident from the definition of J_{p_o} in (2.46).

We summarize what we have just proved in the following theorem.

Theorem 2.3.5. Let $I \subseteq \mathbb{R}$ be an open interval, and $U \subseteq \mathbb{R}^N$ be an open set. Suppose that the vector field $F\colon I \times U \to U$ satisfies a local Lipschitz condition in the second variable at every $(t, p) \in I \times U$. For each $(t_o, p_o) \in I \times U$, there exists a unique C^1 function

$$u_{p_o}\colon J_{p_o} \to U$$

which solved the IVP (2.2) over a maximal interval of existence, J_{p_o}.

Definition 2.3.6. Let $u\colon J_o \to U$ be a solution to the IVP (2.2) defined in some open interval, J_o, which contains t_o. Let J be an open interval which contains t_o and such that $J_o \subset J$. We say that a C^1 curve

$$v\colon J \to U$$

is an extension of u if v solves the IVP (2.2) over the interval J, and

$$v(t) = u(t), \quad \text{for all } t \in J_o.$$

Lemma 2.3.7 (Extensibility Lemma)**.** Let I denote an open interval and U an open subset of $\mathbb{R}^N$. Suppose that $F\colon I \times U \to \mathbb{R}^N$ is continuous and satisfies a local Lipschitz condition in the second variable at every $(t, p) \in I \times U$. Let $(t_o, p_o) \in I \times U$ and $a, b \in \mathbb{R}$ be such that $a < t_o < b$ and $[a, b] \subset I$. Suppose that $u\colon (a, b) \to U$ is a solution of the IVP (2.2) defined on the open interval, (a, b). Then, there exists and extension of u to an interval $(a, b + \delta)$, for some $\delta > 0$, if and only if there exists $p \in U$ and a sequence (t_m) in (a, b) such that

$$\lim_{m\to\infty} (t_m, u(t_m)) = (b, p). \tag{2.48}$$

Proof: If u has an extension to $(a, b + \delta)$, for some $\delta > 0$, the (2.48) holds for any sequence (t_m) which converges to b by the continuity of the extension at b.

In order to prove the converse, assume that (2.48) holds true for a sequence (t_m) in (a, b). Note that it follows from (2.48) that

$$\lim_{m\to\infty} t_m = b, \tag{2.49}$$

and

$$\lim_{m\to\infty} u(t_m) = p. \tag{2.50}$$

We may assume that the sequence (t_m) increases to b, by passing to a subsequence, if necessary.

We first show that

$$\lim_{t\to b^-} u(t) = p. \tag{2.51}$$

Proceeding by contradiction, if (2.51) is not true, there exists $\varepsilon > 0$ and a sequence (s_m) in (a, b) such that (s_m) increases as $m \to \infty$ with

$$\lim_{m\to\infty} s_m = b, \tag{2.52}$$

and

$$\|u(s_m) - p\| \geqslant \varepsilon, \quad \text{for all } m \in \mathbf{N}. \tag{2.53}$$

Passing to a subsequence, if necessary, we may assume that

$$t_m < s_m, \quad \text{for all } m \in \mathbf{N}.$$

Now, let $r_o > 0$ be such that

$$r_o < \varepsilon, \tag{2.54}$$

where $\varepsilon > 0$ is as given in (2.53), and $\overline{B}_{r_o}(p) \subset U$, and let $\delta_o > 0$ be such that $[b - \delta_o, b] \subset I$. Put

$$M_o = \max_{\substack{t\in[b-\delta_o,b]\\ x\in\overline{B}_{r_o}(p)}} \|F(t, x)\|. \tag{2.55}$$

Consider the function

$$g(t) = \|u(t) - p\|, \quad \text{for } t \in (a, b).$$

It follows from (2.50) that, there exits a natural number, N_1, such that

$$g(t_m) < r_o, \quad \text{for all } m \geqslant N_1. \tag{2.56}$$

In view of (2.56), (2.54) and (2.53), it follows from the Intermediate Value Theorem that, for each $m \geqslant N_1$, there exists τ_m such that

$$t_m < \tau_m < s_m, \quad \text{for } m \geqslant N_1, \tag{2.57}$$

and

$$g(\tau_m) = r_o, \quad \text{for } m \geqslant N_1; \tag{2.58}$$

furthermore, we may assume that

$$g(t) \leqslant r_o, \quad \text{for } t \in [t_m, \tau_m], \ \text{ and } m \geqslant N_1. \tag{2.59}$$

It follows from (2.58) and (2.59) that

$$\|u(\tau_m) - p\| = r_o, \quad \text{for } m \geqslant N_1, \tag{2.60}$$

and

$$u(t) \in \overline{B}_{r_o}(p), \quad \text{for } t \in [t_m, \tau_m], \ \text{ for } m \geqslant N_1. \tag{2.61}$$

From the assumption that u solves the IVP (2.2) in (a, b) we obtain that

$$u(t) = p_o + \int_{t_o}^{t} F(\tau, u(\tau)) \ \mathrm{d}\tau, \quad \text{for all } t \in (a, b). \tag{2.62}$$

It follows from (2.62) that

$$u(t_m) = p_o + \int_{t_o}^{t_m} F(\tau, u(\tau)) \ \mathrm{d}\tau, \quad \text{for all } m \in \mathbf{N}, \tag{2.63}$$

and

$$u(\tau_m) = p_o + \int_{t_o}^{\tau_m} F(\tau, u(\tau)) \ \mathrm{d}\tau, \quad \text{for all } m \in \mathbf{N}. \tag{2.64}$$

Subtracting (2.63) from (2.64) we then obtain that

$$u(\tau_m) - u(t_m) = \int_{t_m}^{\tau_m} F(\tau, u(\tau)) \ \mathrm{d}\tau, \quad \text{for all } m \in \mathbf{N}. \tag{2.65}$$

Next, use (2.55) and (2.61) to obtain from (2.65) that

$$\|u(\tau_m) - u(t_m)\| \leqslant M_o|\tau_m - t_m|, \quad \text{for all } m \geqslant N_1. \tag{2.66}$$

Now, use (2.50), (2.52) and (2.57) to derive from (2.66) that

$$\lim_{m\to\infty} \|u(\tau_m) - u(t_m)\| = 0. \tag{2.67}$$

Applying the triangle inequality, we obtain

$$\|u(\tau_m) - p\| \leqslant \|u(\tau_m) - u(t_m)\| + \|u(t_m) - p\|, \quad \text{for all } m. \tag{2.68}$$

Thus, combining (2.50), (2.67) and (2.68) yields

$$\lim_{m\to\infty} \|u(\tau_m) - p\| = 0, \tag{2.69}$$

which, in conjunction with (2.60), implies that $r_o = 0$; but this contradicts $r_o > 0$. Hence, (2.51) must be true.

Next, consider the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t, x); \\ \\ x(b) = p. \end{cases} \tag{2.70}$$

By the Local Existence and Uniqueness Theorem (Theorem 2.0.1), there exists $\delta > 0$ and a unique continuous function

$$v\colon [b - \delta, b + \delta] \to U,$$

which solves the IVP (2.70) in the interval $(b - \delta, b + \delta)$. We may choose δ so that

$$\delta K < 1, \tag{2.71}$$

where K is a Lipschitz constant for F over the set

$$[b - \delta_o, b + \delta_o] \times \overline{B}_{r_o}(p);$$

for some $\delta_o > 0$. We may assume that $\delta < \delta_o$.

Next, define a function $\widehat{u}\colon (a, b+\delta) \to U$ as follows

$$\widehat{u}(t) = \begin{cases} u(t) & \text{if } t \in (a,b); \\ v(t) & \text{if } t \in (b-\delta, b+\delta). \end{cases}$$

We claim that $\widehat{u}$ is an extension of u. To prove this claim, we need to show that

$$u(t) = v(t), \qquad \text{for all } t \in (b-\delta, b). \tag{2.72}$$

In order to prove (2.72), define $w\colon [b-\delta, b] \to U$ as follows

$$w(t) = \begin{cases} u(t) - v(t) & \text{if } b-\delta \leqslant t < b; \\ 0 & \text{if } t = b. \end{cases} \tag{2.73}$$

It follows from (2.51), which was proved in the first part of this proof, that w is continuous on $[b-\delta, b]$. By continuity and (2.51), δ can be chosen so that

$$u(t) \in B_{r_o}(p) \text{ and } v(t) \in B_{r_o}(p), \qquad \text{for all } t \in [b-\delta, b]. \tag{2.74}$$

By continuity, there exists $t_1 \in [b-\delta, b]$ such that

$$\|w(t_1)\| = \max_{t \in [b-\delta, b]} \|w(t)\| \equiv M_1. \tag{2.75}$$

Observe that

$$w(t_1) - w(b) = \int_b^{t_1} w'(\tau)\ \mathrm{d}\tau. \tag{2.76}$$

Thus, using (2.73) and the assumption that u and v solve the differential equation in IVP (2.2), we obtain from (2.76) that

$$w(t_1) = \int_b^{t_1} (F(\tau, u(\tau)) - F(\tau, v(\tau)))\ \mathrm{d}\tau. \tag{2.77}$$

Thus, applying the Lipschitz condition to the expression in (2.77), we get

$$\|w(t_1)\| \leqslant K \left| \int_b^{t_1} \|u(\tau) - v(\tau)\|\ \mathrm{d}\tau \right|,$$

or

$$\|w(t_1)\| \leqslant K \left| \int_b^{t_1} \|w(\tau)\|\ \mathrm{d}\tau \right|,$$

from which we get

$$M_1 \leqslant K\delta M_1. \tag{2.78}$$

in view of (2.75). We then see that, by virtue of (2.71), (2.78) leads to a contradiction, unless $M_1 = 0$. Hence, we obtain from (2.75) that $w(t) = 0$ for all $t \in [b-\delta, b]$, which implies (2.72), and the proof of the extensibility lemma is now complete. ■

Remark 2.3.8. We can use a similar argument to the one used in the proof of Lemma 2.3.7 to prove that, under the same assumptions of the Lemma, u has an extension to $(a-\delta, b)$, for some $\delta > 0$, if and only if there exists $q \in U$ and a sequence (s_m) in (a, b) such that

$$\lim_{m\to\infty}(s_m, u(s_m)) = (a, q).$$

Lemma 2.3.7 will help us answer the following question: Suppose that $J_{p_o} = (a, b)$ is the maximal interval of existence for IVP (2.2), for some $(t_o, p_o) \in I \times U$, and let $u\colon J_{p_o} \to U$ denote the corresponding integral curve. If $b < \infty$, what happens to $u(t)$ as t tends to b? We will see, as a consequence of the following proposition, that either $u(t)$ tends to some point on the boundary, ∂U, of U, or $\|u(t)\| \to \infty$ as $t \to b^-$. Example 2.3.1 on page 17 of these notes provides an instance of this general result.

Proposition 2.3.9 (Escape in Finite Time Theorem)**.** Let I denote an open interval and U an open subset of $\mathbb{R}^N$. Suppose that $F\colon I\times U \to \mathbb{R}^N$ is continuous and satisfies a local Lipschitz condition in the second variable at every point $(t, p) \in I \times U$. Let $J_{p_o} = (a, b)$ denote the maximal interval of existence for IVP (2.2), for some $(t_o, p_o) \in I \times U$, and let $u\colon J_{p_o} \to U$ be the corresponding integral curve. If $b < \infty$, then, for any compact set, $C \subset U$, there exists $\varepsilon > 0$ such that

$$t \in (b-\varepsilon, b) \Rightarrow u(t) \notin C.$$

Proof: Suppose, to the contrary, that there exists a compact set, $C \subset U$, and a sequence, (t_m), in (a, b) such that t_m tends to b as $m \to \infty$ and $u(t_m) \in C$ for all $m \in \mathbf{N}$. Then, since $[t_o, b] \times C$ is compact, passing to a subsequence if necessary, we may assume that $(t_m, u(t_m))$ converges (b, p), for some $p \in C$; that is,

$$\lim_{m\to\infty}(t_m, u(t_m)) = (b, p).$$

Hence, the hypotheses of Lemma 2.3.7 are satisfied. Consequently, there exists an extension of u to $(a, b+\delta)$, for some $\delta > 0$. However, this contradicts the maximality of J_{p_o}. Thus, the proposition follows. ■

Remark 2.3.10. Under the same assumptions of Proposition 2.3.9, we may prove that if $a > -\infty$, then, for any compact set, $C \subset U$, there exists $\varepsilon > 0$, such that $t \in (a, a+\varepsilon) \Rightarrow u(t) \notin C$.

Remark 2.3.11. Proposition 2.3.9 is very useful in verifying that a solution of IVP (2.2) exists for all $t \geqslant t_o$ for the case in which $F(t, x)$ is known to be continuous over $I \times \mathbb{R}^N$, where I is an open interval containing $[t_o, \infty)$, and we can obtain an *a priori* estimate on the norm, $\|u(t)\|$, for the solution u on IVP (2.2) over the maximal interval of existence $J_{p_o} = (a, b)$. More specifically, arguing by contradiction, suppose that $b < \infty$, and that we can, *a priori*, obtain the estimate

$$\|u(t)\| \leqslant R, \quad \text{for all } t \in [t_o, b), \tag{2.79}$$

and some $R > 0$. We can then consider the compact set

$$C = \{x \in \mathbb{R}^N \mid \|x\| \leqslant R\}.$$

It follows from (2.79) that $u(t) \in C$ for all $t \in [t_o, b)$, which is in direct contradiction with the result of Proposition 2.3.9. In the remainder of this section, we will see a few applications of this idea.

For the remainder of this section, assume that $I = \mathbb{R}$ and $U = \mathbb{R}^N$, so that the vector field,

$$F \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N,$$

is defined for all times, $t \in \mathbb{R}$. We will ask the question: Under which conditions on the field, F, is $J_{p_o} = \mathbb{R}$? That is, under which conditions on F is the solution to the IVP (2.2) defined for all times $t \in \mathbb{R}$? We will see that the following growth condition on the vector field $F(t,x)$ is sufficient:

Definition 2.3.12 (At Most Linear Growth)**.** Let $F \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous vector field. We say that $F(t,x)$ has at most linear growth in the second variable if and only if, for every $T > 0$, there exist non–negative constants, c_T and d_T, such that

$$\|F(t,x)\| \leqslant c_T + d_T\|x\|, \quad \text{for all } t \in [-T,T], \text{ and all } x \in \mathbb{R}^N. \tag{2.80}$$

Proposition 2.3.13 (Global Existence Theorem I)**.** Let $F \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a locally Lipschitz continuous vector field. Suppose that $F(t,x)$ has at most linear growth in the second variable. Then, for any $(t_o, p_o) \in \mathbb{R} \times \mathbb{R}^N$, the solution to the IVP (2.2) exists for all $t \in \mathbb{R}$; in other words, the maximal interval of existence, J_{p_o}, is the entire real line.

Proof: Assume, by way of contradiction, that $J_{p_o} = (a,b)$ is such that $b < \infty$ (the case in which $a > -\infty$ can be treated in an analogous manner). By the growth assumption in (2.80), there exist non–negative constants, c_b and d_b, such that

$$\|F(t,x)\| \leqslant c_b + c_d\|x\|, \quad \text{for all } t \in [t_o, b] \text{ and all } x \in \mathbb{R}^n. \tag{2.81}$$

Let $u \colon (a,b) \to \mathbb{R}^N$ be the solution to the IVP (2.2). We can then write

$$u(t) = p_o + \int_{t_o}^{t} F(\tau, u(\tau))\ \mathrm{d}\tau, \text{ for all } t \in (a,b). \tag{2.82}$$

Using the estimate in (2.82), we get from (2.82) that

$$\|u(t)\| \leqslant \|p_o\| + \int_{t_o}^{t} (c_b + d_b\|u(\tau)\|)\ \mathrm{d}\tau, \text{ for all } t \in [t_o, b),$$

from which we get

$$\|u(t)\| \leqslant \|p_o\| + c_b(b - t_o) + d_b \int_{t_o}^{t} \|u(\tau)\|\ \mathrm{d}\tau, \text{ for all } t \in [t_o, b). \tag{2.83}$$

Putting $C_o = \|p_o\| + c_b(b - t_o)$ in the inequality in (2.83) yields

$$\|u(t)\| \leqslant C_o + d_b \int_{t_o}^{t} \|u(\tau)\| \, \mathrm{d}\tau, \text{ for all } t \in [t_o, b). \tag{2.84}$$

Define $g\colon [t_o, b) \to \mathbb{R}$ by

$$g(t) = C_o + d_b \int_{t_o}^{t} \|u(\tau)\| \, \mathrm{d}\tau, \quad \text{ for } t \in [t_o, b). \tag{2.85}$$

It follows from the Fundamental Theorem of Calculus and (2.85) that g is differentiable on (t_o, b) with

$$g'(t) = d_b \|u(t)\|, \quad \text{ for } t \in (t_o, b). \tag{2.86}$$

Using the estimate in (2.84), we obtain from (2.86)

$$g'(t) \leqslant d_b \; g(t), \quad \text{ for } t \in (t_o, b), \tag{2.87}$$

where we have also used (2.85). Rewrite the differential inequality in (2.87) as

$$g'(t) - d_b \; g(t) \leqslant 0, \quad \text{ for } t \in (t_o, b), \tag{2.88}$$

and multiply on both sides of (2.88) by $e^{-d_b t}$ to obtain

$$\frac{d}{dt}\left[e^{-d_b t} g(t)\right] \leqslant 0, \quad \text{ for } t \in (t_o, b). \tag{2.89}$$

Integrating on both sides of the inequality in (2.89) from t_o to t leads to

$$g(t) \leqslant g(t_o) e^{d_b(t - t_o)}, \quad \text{ for } t \in (t_o, b). \tag{2.90}$$

Consequently, using (2.84) and (2.85), we get from (2.90) that

$$\|u(t)\| \leqslant C_o e^{d_b(b - t_o)}, \quad \text{ for } t \in [t_o, b). \tag{2.91}$$

Setting $R_o = C_o e^{d_b(b-t_o)}$, we see from (2.91) that the values $u(t)$ lie in the compact set $\overline{B}_{R_o}(0)$. This is in direct contradiction with the result of Proposition 2.3.9. Hence, $b = \infty$. ■

Example 2.3.14 (Linear Systems)**.** Let $f\colon \mathbb{R} \to \mathbb{R}^N$ be a continuous function, and let

$$a_{ij}\colon \mathbb{R} \to \mathbb{R}, \quad \text{ for } i, j = 1, 2, \ldots, N,$$

be continuous real valued functions. Define the matrix valued function

$$A(t) = [a_{ij}(t)], \quad \text{ for } t \in \mathbb{R}.$$

The vector field, $F\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, defined by

$$F(t, x) = f(t) + A(t)x, \quad \text{ for all } t \in \mathbb{R} \text{ and all } x \in \mathbb{R}^N, \tag{2.92}$$

satisfies the growth condition in (2.80). In fact, we have the estimate

$$\|F(t,x)\| \leqslant \|f(t)\| + \|A(t)\|\|x\|, \quad \text{for all } t \in \mathbb{R} \text{ and all } x \in \mathbb{R}^N, \tag{2.93}$$

where

$$\|A(t)\| = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}[a_{ij}(t)]^2}, \quad \text{for all } t \in \mathbb{R}.$$

The growth estimate in (2.80) then follows from (2.93) and the assumption that f and a_{ij}, for $i, j = 1, 2, \ldots, N$, are continuous functions.

The local Lipschitz continuity of F in (2.92) follows from the estimate

$$\|F(t,x) - F(t,y)\| \leqslant \|A(t)\|\|x - y\|, \quad \text{for all } t \in \mathbb{R} \text{ and all } x, y \in \mathbb{R}^N,$$

and the continuity of a_{ij}, for $i, j = 1, 2, \ldots, N$.

We can then conclude, as a consequence of Proposition 2.3.13 that the linear IVP

$$\begin{cases} \dfrac{dx}{dt} = f(t) + A(t)x; \\ \\ x(t_o) = p_o. \end{cases} \tag{2.94}$$

has a unique solution for each $(t_o, p_o) \in \mathbb{R} \times \mathbb{R}^N$, which exists for all $t \in \mathbb{R}$. Later in these notes we will see how to compute the solution of the linear IVP (2.94).

Corollary 2.3.15 (Global Existence Theorem II)**.** Let $F\colon \mathbb{R}^N \to \mathbb{R}^N$ satisfy the global Lipschitz condition:

$$\|F(x) - F(y)\| \leqslant K\|x - y\|, \quad \text{for } x, y \in \mathbb{R}^N. \tag{2.95}$$

Then, for any $(t_o, p_o) \in \mathbb{R} \times \mathbb{R}^N$, the solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(t_o) = p_o. \end{cases} \tag{2.96}$$

exists for all $t \in \mathbb{R}$.

Proof: Observe that, for any $x \in \mathbb{R}^N$, it follows from (2.95) that

$$\|F(x) - F(p_o)\| \leqslant K\|x - p_o\|,$$

from which we get that

$$\|F(x)\| \leqslant \|F(p_o)\| + \|p_o\| + K\|x\|.$$

Hence, F satisfies a growth condition of the type in (2.80) needed for Proposition 2.3.13. The corollary then follows from the Global Existence I theorem (Proposition 2.3.13). ■

Corollary 2.3.16 (Global Existence Theorem III)**.** Let $F\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be continuous and bounded. Suppose that $F(t,x)$ satisfies a local Lipschitz condition in the second variable at every point $(t,p)\in\mathbb{R}\times\mathbb{R}^N$. Then, for any $(t_o,p_o)\in\mathbb{R}\times\mathbb{R}^N$, the solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ \\ x(t_o) = p_o. \end{cases} \tag{2.97}$$

exists for all $t\in\mathbb{R}$.

2.4 Continuous Dependence on Initial Conditions

Let $U\subseteq\mathbb{R}^N$ be an open set and I denote an open interval. Assume that $F\colon I\times U\to\mathbb{R}^N$ is continuous and that $F(t,x)$ satisfies a local Lipschitz condition in the second variable at every point $(t,p)\in I\times U$. In the previous section we saw that, for $t_o\in I$ and $p\in U$, the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ \\ x(t_o) = p, \end{cases} \tag{2.98}$$

has a unique solution,

$$u_p\colon J_p\to U,$$

defined on a maximal interval of existence, J_p, containing t_o. Similarly, for $q\in U$, we get a unique solution,

$$u_q\colon J_q\to U,$$

of the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ \\ x(t_o) = q, \end{cases} \tag{2.99}$$

defined on a maximal interval of existence, J_q, containing t_o. We would like to answer the following question in this section: Suppose that p and q are close to each other; that is, $\|p-q\|$ is small. How close will the values of, $u_p(t)$ and $u_q(t)$, of u_p and u_q, respectively, be to each other over an interval $[t_o,t_1]$, for some $t_1>t_o$ with $t_1\in J_p\cap J_q$? The idea here is that if $u_p(t_o)$ and $u_q(t_o)$ start out being very close to each other, then the remaining values, $u_p(t)$ and $u_q(t)$ for $t\in[t_o,t_1]$, will remain close to one another.

Let B denote an open ball containing p and q be such that $\overline{B} \subset U$ and

$$\|F(t,x) - F(t,y)\| \leqslant K\|x-y\|, \quad \text{for } x, y \in \overline{B}, \text{ and } |t-t_o| \leqslant \delta_o, \tag{2.100}$$

for positive constants K and δ_o. We assume that t_1 is sufficiently close to t_o so that

$$[t_o, t_1] \subset (t_o - \delta_o, t_o + \delta_o), \tag{2.101}$$

and

$$u_p(t), u_q(t) \in B. \quad \text{for all } t \in [t_o, t_1]. \tag{2.102}$$

Using the integral representation for the solutions of the IVPs in (2.98) and (2.99), we have that

$$u_p(t) = p + \int_{t_o}^{t} F(\tau, u_p(\tau))\, \mathrm{d}\tau, \quad \text{for all } t \in J_p, \tag{2.103}$$

and

$$u_q(t) = q + \int_{t_o}^{t} F(\tau, u_q(\tau))\, \mathrm{d}\tau, \quad \text{for all } t \in J_q. \tag{2.104}$$

Subtracting (2.104) from (2.103) we obtain that

$$u_p(t) - u_q(t) = p - q + \int_{t_o}^{t} (F(\tau, u_p(\tau)) - F(\tau, u_q(\tau)))\, \mathrm{d}\tau, \tag{2.105}$$

for all $t \in [t_o, t_1]$. Taking the Euclidean norm on both sides of (2.105) we obtain the estimate

$$\|u_p(t) - u_q(t)\| \leqslant \|p-q\| + \int_{t_o}^{t} \|F(\tau, u_p(\tau)) - F(\tau, u_q(\tau))\|\, \mathrm{d}\tau, \tag{2.106}$$

for all $t \in [t_o, t_1]$.

Next, use (2.102), (2.101) and the Lipschitz condition in (2.100) to obtain from (2.106) that

$$\|u_p(t) - u_q(t)\| \leqslant \|p-q\| + K\int_{t_o}^{t} \|u_p(\tau) - u_q(\tau)\|\, \mathrm{d}\tau, \tag{2.107}$$

for all $t \in [t_o, t_1]$. Next, set

$$g(t) = \|p-q\| + K\int_{t_o}^{t} \|u_p(\tau) - u_q(\tau)\|\, \mathrm{d}\tau, \quad \text{for } t \in [t_o, t_1]. \tag{2.108}$$

It follows from (2.107) and (2.108) that

$$\|u_p(t) - u_q(t)\| \leqslant g(t), \quad \text{for } t \in [t_o, t_1]. \tag{2.109}$$

Applying the Fundamental Theorem of Calculus to the definition of g in (2.108) we see that g is differentiable on (t_o, t_1), and

$$g'(t) = K\|u_p(t) - u_q(t)\|, \quad \text{for } t \in (t_o, t_1). \tag{2.110}$$

Combining (2.110) with the estimate in (2.109), we obtain that g satisfies the differential inequality

$$g'(t) \leqslant Kg(t), \quad \text{for } t \in (t_o, t_1),$$

which can be rewritten as

$$g'(t) - Kg(t) \leqslant 0, \quad \text{for } t \in (t_o, t_1). \tag{2.111}$$

Multiplying the inequality in (2.111) by e^{-Kt} leads to

$$\frac{d}{dt}\left[e^{-Kt}g(t)\right] \leqslant 0, \quad \text{for } t \in (t_o, t_1). \tag{2.112}$$

Integrating on both sides of (2.112) from t_o to $t \in (t_o, t_1)$ then leads to

$$e^{-Kt}g(t) \leqslant g(t_o)e^{-Kt_o}, \quad \text{for } t \in (t_o, t_1), \tag{2.113}$$

where

$$g(t_o) = \|p - q\|, \tag{2.114}$$

by virtue of (2.108). We then obtain from (2.113) and (2.114) that

$$g(t) \leqslant \|p - q\|e^{K(t-t_o)}, \quad \text{for } t \in (t_o, t_1). \tag{2.115}$$

Combining (2.109) and (2.115) then yields

$$\|u_p(t) - u_q(t)\| \leqslant \|p - q\|e^{K(t-t_o)}, \quad \text{for } t \in [t_o, t_1]. \tag{2.116}$$

It follows from (2.116) that

$$\max_{t\in[t_o,t_1]} \|u_p(t) - u_q(t)\| \leqslant C_1\|p - q\|, \tag{2.117}$$

where $C_1 = e^{K(t_1-t_o)}$.

Thus, we get from (2.117) that, given $\varepsilon > 0$, there exists $r > 0$ such that

$$\|p - q\| < r \Rightarrow d(u_p, u_q) < \varepsilon, \tag{2.118}$$

where

$$d(u_p, u_q) = \max_{t\in[t_o,t_1]} \|u_p(t) - u_q(t)\|. \tag{2.119}$$

In fact, (2.118) follows from (2.117) and (2.119) by choosing $r = \varepsilon/C_1$. Another way of expressing what we have just proved is

$$\lim_{p\to q} d(u_p, u_q) = 0,$$

which says that the solutions, u_p, of IVP (2.98) depends continuously on the initial conditions, p, for a sufficiently small interval $[t_o, t_1]$.

We would like to extend the local continuity result we just proved to larger closed and bounded intervals on which a solution exists. More precisely,

Theorem 2.4.1 (Continuous Dependence on Initial Conditions)**.** Let I denote an open interval and U an open subset of $\mathbb{R}^N$. Suppose that $F\colon I \times U \to \mathbb{R}^N$ is continuous and satisfies a local Lipschitz condition in the second variable at every point $(t,p) \in I \times U$. Let $(t_o, p_o) \in I \times U$ and let J_{p_o} denote the maximal interval of existence for the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ \\ x(t_o) = p_o. \end{cases} \tag{2.120}$$

For every real number, T, such that $T > t_o$ and $[t_o, T] \subset J_{p_o}$, there exists $r(T) > 0$, such that, if

$$\|p - p_o\| \leqslant r(T),$$

then the solution, u_p, of the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ \\ x(t_o) = p, \end{cases} \tag{2.121}$$

exists on $[t_o, T]$. Furthermore,

$$\lim_{p\to p_o} \max_{t\in[t_o,T]} \|u_p(t) - u_{p_o}(t)\| = 0, \tag{2.122}$$

where $u_{p_o}\colon J_{p_o} \to U$ is the solution to the IVP (2.120).

Proof: Let $T > t_o$ be such that $[t_o, T] \subset J_{p_o}$. Since $u_{p_o}\colon J_{p_o} \to U$ is continuous, the set

$$C = \{u_{p_o}(t) \mid t_o \leqslant t \leqslant T\}$$

is a compact subset of U. Thus, there exists an open set, V, such that $C \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. We first show that there exists a constant, $K > 0$, such that

$$\|F(t,x) - F(t,y)\| \leqslant K\|x - y\|, \quad \text{for } x, y \in \overline{V}, \text{ and } \ t \in [t_o, T]. \tag{2.123}$$

To prove (2.123), assume, by way of contradiction, that there exist sequences (t_m) in $[t_o, T]$, and (x_m) and (y_m) in $\overline{V}$, such that

$$\|F(t_m, x_m) - F(t_m, y_m)\| > m\|x_m - y_m\|, \qquad \text{for } m \in \mathbf{N}. \tag{2.124}$$

Using the fact that $[t_o, T] \times \overline{V}$ is compact, we may assume, passing to subsequences if necessary, that there exist $\overline{t} \in [t_o, T]$ and $\overline{x}, \overline{y} \in \overline{V}$ such that

$$\lim_{m\to\infty} t_m = \overline{t}, \tag{2.125}$$

$$\lim_{m\to\infty} x_m = \overline{x}, \tag{2.126}$$

and

$$\lim_{m\to\infty} y_m = \overline{y}. \tag{2.127}$$

Let

$$M = \max_{(t,x)\in[t_o,T]\times\overline{V}} \|F(t,x)\|. \tag{2.128}$$

It follows from (2.124) and (2.128) that

$$\|x_m - y_m\| \leqslant \frac{2M}{m}, \tag{2.129}$$

where we have used the triangle inequality.

Letting $m \to \infty$ in (2.129), we obtain from (2.126) and (2.127) that

$$\|\overline{x} - \overline{y}\| = 0.$$

Hence, $\overline{x} = \overline{y}$. Put $q = \overline{x} = \overline{y}$, and let $r_1 > 0$, $\delta_1 > 0$ and K_1 be such that

$$[\overline{t} - \delta_1, \overline{t} + \delta_1] \subset I,$$

$$\overline{B}_{r_1}(q) \subset U,$$

and

$$\|F(t,x) - F(t,y)\| \leqslant K_1\|x - y\|, \quad \text{for } x, y \in \overline{B}_{r_1}(q), \text{ and } |t - \overline{t}| \leqslant \delta_1. \tag{2.130}$$

By virtue of (2.125), (2.126) and (2.127), there exists $N_1 > K_1$ such that

$$m \geqslant N_1 \Rightarrow x_m, y_m \in B_{r_1}(q), \quad \text{and } t_m \in (\overline{t} - \delta_1, \overline{t} + \delta_1).$$

It then follows from (2.130) that

$$\|F(t_m, x_m) - F(t_m, y_m)\| \leqslant m\|x_m - y_m\|, \quad \text{for all } m \geqslant N_1. \tag{2.131}$$

However, (2.131) is in direct contradiction with (2.124). We therefore conclude that (2.123) must be true.

Next, let

$$\varepsilon = \frac{1}{2}\operatorname{dist}(C, \partial V), \tag{2.132}$$

so that $\varepsilon > 0$, and define

$$r(T) = \varepsilon e^{-K(T-t_o)}, \tag{2.133}$$

where K is the Lipschitz constant given in (2.123). Put

$$C_\varepsilon = \{x \in V \mid \operatorname{dist}(x, C) \leqslant \varepsilon\}. \tag{2.134}$$

Then, C_ε is a compact subset of V which contains C.

We claim that if $\|p - p_o\| < r(T)$, then the solution,

$$u_p \colon J_p \to U,$$

of IVP (2.121) is defined on $[t_o, T]$. If not, writing $J_p = (a, b)$, we would have that $b < T$. It then follows from the Escape in Finite Time Theorem (Proposition 2.3.9) that there exists $t_1 \in [t_o, b)$ such that $u_p(t_1) \notin C_\varepsilon$, where C_ε is defined in (2.134). We may also assume that

$$u_p(t) \in V \quad \text{for all } t \in [t_o, t_1]; \tag{2.135}$$

we also have that

$$\|u_p(t_1) - u_{p_o}(t_1)\| > \varepsilon. \tag{2.136}$$

Now, the calculations leading to (2.116) yield that

$$\|u_p(t) - u_{p_o}(t)\| \leqslant \|p - p_o\| e^{K(t-t_o)}, \quad \text{for } t \in [t_o, t_1],$$

in view of (2.135) and the Lipschitz condition in (2.123); so that,

$$\|u_p(t) - u_{p_o}(t)\| \leqslant \|p - p_o\| e^{K(T-t_o)}, \quad \text{for } t \in [t_o, t_1]. \tag{2.137}$$

Thus, if $\|p - p_o\| < r(T)$, where $r(T)$ is given in (2.133), we have from the inequality in (2.137) that

$$\|u_p(t) - u_{p_o}(t)\| \leqslant \varepsilon, \quad \text{for } t \in [t_o, t_1],$$

where $\varepsilon > 0$ is given in (2.132), which is in direct contradiction with (2.136). Hence, $\|p - p_o\| < r(T)$ implies that u_p is defined on $[t_o, T]$. Furthermore, we get the estimate

$$\|u_p(t) - u_{p_o}(t)\| \leqslant \|p - p_o\| e^{K(T-t_o)}, \quad \text{for } t \in [t_o, T]. \tag{2.138}$$

The statement in (2.122) follows from (2.138), and the proof of the theorem is now complete. ■

Chapter 3

Flows of C^1 Vector Fields

In this chapter we consider the autonomous system

$$\frac{dx}{dt} = F(x), \tag{3.1}$$

where $F\colon U \to \mathbb{R}^N$ is a C^1 vector field defined on some open set $U \subseteq \mathbb{R}^N$. By the fundamental theory presented in the previous chapter, we know that for each $p \in U$, there is a unique integral curve,

$$u_p\colon J_p \to U,$$

defined on a maximal interval of existence, J_p, and which solves the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p. \end{cases} \tag{3.2}$$

We can therefore define a map

$$\theta\colon \mathcal{D} \to U,$$

on a subset, $\mathcal{D}$, of $\mathbb{R} \times U$ given by

$$\mathcal{D} = \{(t,p) \in \mathbb{R} \times U \mid t \in J_p\}, \tag{3.3}$$

as follows

$$\theta(t,p) = u_p(t), \qquad \text{for all } (t,p) \in \mathcal{D}. \tag{3.4}$$

One of the goals of this chapter is to give a precise definition of the map θ given by (3.4). In particular, we will be showing that the domain of θ, $\mathcal{D}$, given in (3.3) is an open subset of $\mathbb{R} \times U$. We will also be showing that θ is continuous. We will then derive some fundamental properties of θ. The set $\mathcal{D}$ is called the flow domain of the vector field, F, and θ is the flow of F.

3.1 Flow Domains and Flow Maps

Throughout this section, $F\colon U \to \mathbb{R}^N$ will denote a C^1 vector field defined in an open subset, U, of $\mathbb{R}^N$. For each $p \in U$, $u_p\colon J_p \to U$ will denote the unique solution to the IVP (3.2) defined on a maximal interval of existence, J_p, which contains 0. Define

$$\mathcal{D} = \{(t,p) \in \mathbb{R} \times U \mid t \in J_p\}. \tag{3.5}$$

Proposition 3.1.1. The set $\mathcal{D}$ defined in (3.5) is an open subset of $\mathbb{R} \times U$.

Proof: Let $(t_1, p_1) \in \mathcal{D}$. We show that there exists $\delta_1 > 0$ and an open ball $B = B_{r_1}(p_1)$ satisfying

$$(t_1 - \delta_1, t_1 + \delta_1) \times B \subset \mathcal{D}. \tag{3.6}$$

Since $(t_1, p_1) \in \mathcal{D}$, there exists

$$u_{p_1}\colon J_{p_1} \to U$$

which solves the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p_1; \end{cases} \tag{3.7}$$

furthermore, $t_1 \in J_{p_1}$.

Assume first that $t_1 > 0$. Then u_{p_1} is defined on $[0, t_1]$. Consequently, by the Extensibility Lemma (Lemma 2.3.7), u_{p_1} is also defined on $[0, t_1 + \delta_1]$, for some $\delta_1 > 0$. By the Continuous Dependence on Initial Conditions Theorem (Theorem 2.4.1 on page 30 in this notes), there exists $r_1 > 0$ such that $\|p - p_1\| < r_1$ implies that the solution of

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p, \end{cases} \tag{3.8}$$

is defined on $[0, t_1 + \delta_1]$. Thus, setting $B = B_{r_1}(p_1)$ we see that if $(t,p) \in (t_1 - \delta_1, t_1 + \delta_1) \times B$, then $u_p\colon J_p \to U$ is defined on $[0, t_1 + \delta_1]$, so that $t \in J_p$, and so $(t,p) \in \mathcal{D}$. The statement in (3.6) then follows for the case in which $t_1 > 0$. The case in which $t_1 < 0$ is analogous. We can therefore conclude that $\mathcal{D}$ is open. ■

We may define a map, $\theta\colon \mathcal{D} \to U$, on the flow domain $\mathcal{D}$ as follows:

Definition 3.1.2 (Flow Map of a Vector Field)**.** For each $(t,p) \in \mathcal{D}$, let $u_p\colon J_p \to U$ denote the unique solution to the IVP (3.7) defined on a maximal interval of existence, J_p. Put

$$\theta(t,p) = u_p(t), \quad \text{for } (t,p) \in \mathcal{D}. \tag{3.9}$$

The map $\theta\colon \mathcal{D} \to U$ is called the flow map of the vector field F. Equivalently, θ is called the flow map of the differential equation

$$\frac{dx}{dt} = F(x). \tag{3.10}$$

3.2 Properties of Flow Maps

The following proposition shows that the flow map, $\theta\colon \mathcal{D} \to U$, given in Definition 3.1.2 is continuous on the flow domain, $\mathcal{D}$.

Proposition 3.2.1. The flow map, $\theta\colon \mathcal{D} \to U$, is continuous on $\mathcal{D}$.

Proof: Let $(t_1, p_1) \in \mathcal{D}$. We prove that

$$\lim_{(t,p)\to(t_1,p_1)} \|\theta(t,p) - \theta(t_1,p_1)\| = 0. \tag{3.11}$$

Let $u_{p_1}\colon J_{p_1} \to U$ be the unique solution of the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p_1; \end{cases} \tag{3.12}$$

defined over a maximal interval of existence, J_{p_1}. Since $(t_1, p_1) \in \mathcal{D}$, we know that $t_1 \in J_{p_1}$. Assume first that $t_1 > 0$; the case $t_1 < 0$ is analogous. Then u_{p_1} is defined on $[0, t_1]$. Consequently, by the Extensibility Lemma (Lemma 2.3.7), u_{p_1} is also defined on $[0, t_1 + \delta_1]$, for some $\delta_1 > 0$. By the Continuous Dependence on Initial Conditions Theorem (Theorem 2.4.1 on page 30 in these notes), there exists $r_1 > 0$ such that $\|p - p_1\| < r_1$ implies that the solution of

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p, \end{cases} \tag{3.13}$$

is defined on $[0, t_1 + \delta_1]$. The calculations leading to (2.137) in the proof of Theorem 2.4.1 yield that

$$\|u_p(t) - u_{p_1}(t)\| \leqslant \|p - p_1\| e^{Kt}, \quad \text{for all } t \in [0, t_1 + \delta_1], \tag{3.14}$$

and for some constant K, provided that $\|p - p_1\| < r_1$. Thus, if $\|p - p_1\| < r_1$, we obtain from (3.14) that

$$\|u_p(t) - u_{p_1}(t)\| \leqslant \|p - p_1\| e^{K(t_1+\delta_1)}, \quad \text{for all } t \in [0, t_1 + \delta_1]. \tag{3.15}$$

Next, let $|t - t_1| < \delta_1$ and $\|p - p_1\| < r_1$ and apply the triangle inequality in

$$\|\theta(t,p) - \theta(t_1,p_1)\| = \|\theta(t,p) - \theta(t,p_1) + \theta(t,p_1) - \theta(t_1,p_1)\|$$

to obtain

$$\|\theta(t,p)-\theta(t_1,p_1)\| \leqslant \|u_p(t)-u_{p_1}(t)\| + \|u_{p_1}(t)-u_{p_1}(t_1)\|, \tag{3.16}$$

for $|t-t_1|<\delta_1$ and $\|p-p_1\|<r_1$, where we have also used the definition of the flow map in (3.9). It then follows from (3.16) and (3.15) that

$$\|\theta(t,p)-\theta(t_1,p_1)\| \leqslant C_1\|p-p_1\| + \|u_{p_1}(t)-u_{p_1}(t_1)\|, \tag{3.17}$$

for $|t-t_1|<\delta_1$ and $\|p-p_1\|<r_1$, where $C_1 = e^{K(t_1+\delta_1)}$. The statement in (3.11) now follows from (3.17) and the fact that the function u_{p_1} is continuous on J_{p_1}. The proof of Proposition 3.2.1 is now complete. ■

For the proof of Proposition 3.2.1 all that we needed to assume is that F be locally Lipschitz continuous. For a C^1 vector field, F, we'll be able to prove more than was proved in Proposition 3.2.1. We will show next that, for C^1 vector fields, the flow map, $\theta\colon \mathcal{D}\to U$, is a C^1 map; that is, the partial derivatives

$$\frac{\partial\theta}{\partial t},\ \frac{\partial\theta}{\partial x_i}, \qquad \text{for } i=1,2,\ldots,n,$$

where $x_1,x_2,\ldots,x_n$ are the coordinates of the initial points, p, in the IVP (3.13), are continuous on $\mathcal{D}$.

Proposition 3.2.2. Assume that the vector field, $F\colon U\to\mathbb{R}^N$, is C^1. Then, the flow map, $\theta\colon\mathcal{D}\to U$, is C^1 on $\mathcal{D}$.

Before presenting a proof of Proposition 3.2.2, we will first illustrate its result in one dimension. We will also introduce some notation that will be useful later in these notes.

Example 3.2.3 (One–dimensional Flow Map)**.** Let U denote an open subset in $\mathbb{R}$ and $f\colon U\to\mathbb{R}$ be a real-valued C^1 map. Consider the first order ODE

$$\frac{dx}{dt} = f(x). \tag{3.18}$$

Let $\mathcal{D}\subset\mathbb{R}\times U$ denote the flow domain for f and $\theta(t,p)$, for $(t,p)\in\mathcal{D}$, be the corresponding flow map. For fixed $t\in\mathbb{R}$, we consider the map

$$p\mapsto\theta(t,p).$$

We will denote this map by $\theta_t\colon U\to U$; thus,

$$\theta_t(p)=\theta(t,p), \qquad \text{for all } p\in U \text{ with } t\in J_p.$$

We will show that θ_t is differentiable at any p in U; that is, we will show that

$$\lim_{h\to 0}\frac{\theta_t(p+h)-\theta_t(p)}{h} \quad \text{exists.} \tag{3.19}$$

Suppose for the moment that the statement in (3.19) is true and denote the limit in (3.19) by $v(t,p)$. Assume also that v is continuous. Now, since the map $t \mapsto \theta(t,p)$ solves the differential equation in (3.18) it follows that

$$\frac{\partial \theta}{\partial t}(t,p) = f(\theta(t,p)) \tag{3.20}$$

for all $(t,p) \in \mathcal{D}$. Differentiating the expression in (3.20) with respect to p, and applying the Chain Rule on the right–hand side of (3.20), we obtain that

$$\frac{\partial}{\partial t}\left[\frac{\partial \theta}{\partial p}(t,p)\right] = f'(\theta(t,p))\frac{\partial \theta}{\partial p}(t,p), \tag{3.21}$$

where we have interchanged the order of differentiation on the left–hand side of (3.21). It follows from (3.21) that, for fixed p,

$$v_p(t) \equiv v(t,p) = \frac{\partial \theta}{\partial p}(t,p) \tag{3.22}$$

solves the linear differential equation

$$\frac{dy}{dt} = a_p(t)y, \tag{3.23}$$

where

$$a_p(t) = f'(\theta(t,p)), \qquad \text{for } t \in J_p. \tag{3.24}$$

From (3.22) and the fact that

$$\theta(0,p) = p, \qquad \text{for all } p \in U,$$

we obtain that

$$v_p(0) = 1. \tag{3.25}$$

Consequently, v_p solve the linear IVP

$$\begin{cases} \dfrac{dy}{dt} = a_p(t)y; \\ \\ y(0) = 1. \end{cases} \tag{3.26}$$

Motivated by the previous observations, we now prove that (3.19) holds true; in fact,

$$\lim_{h \to 0} \frac{\theta_t(p+h) - \theta_t(p)}{h} = v(t,p), \tag{3.27}$$

where $v_p(t) = v(t,p)$ is the solution to the IVP (3.26), where $a_p(t)$ is given by (3.24).

Assume first that $t > 0$ and let $T > t$ be such that $[0,T] \subset J_p$. By the Continuous Dependence on Initial Conditions Theorem (Theorem 2.4.1 on page 30 of these notes), there exists $\delta = \delta(T) > 0$ such that

$$|h| < \delta \Rightarrow p + \delta \in U,$$

and $\theta(t,p+h)$ is defined on $[0,T]$. Define

$$g(t,h) = \theta(t,p+h) - \theta(t,p) - v(t,p)h, \quad \text{for } t \in [0,T], \text{ and } |h| < \delta. \tag{3.28}$$

We show that, for each $t \in [0,T]$,

$$g(t,h) = o(h), \quad \text{as } |h| \to 0; \tag{3.29}$$

that is,

$$\lim_{|h|\to 0} \frac{|g(t,h)|}{|h|} = 0.$$

This will prove (3.27).

Write

$$\theta(t,p) = p + \int_0^t f(\theta(\tau,p))\, \mathrm{d}\tau, \tag{3.30}$$

$$\theta(t,p+h) = p + h + \int_0^t f(\theta(\tau,p+h))\, \mathrm{d}\tau, \tag{3.31}$$

and

$$v(t,p) = 1 + \int_0^t f'(\theta(\tau,p))v(\tau,p)\, \mathrm{d}\tau, \tag{3.32}$$

for $t \in [0,T]$ and $|h| < \delta$. Substituting (3.30)–(3.32) into (3.28) then yields

$$g(t,h) = \int_0^t [f(\theta(\tau,p+h)) - f(\theta(\tau,p)) - f'(\theta(\tau,p))v(\tau,p)h]\, \mathrm{d}\tau, \tag{3.33}$$

for $t \in [0,T]$ and $|h| < \delta$.

Now, using the assumption that f is differentiable on U, we can write

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + R(x, \Delta x), \tag{3.34}$$

where

$$R(x,\Delta x) = o(\Delta x), \quad \text{as } |\Delta x| \to 0. \tag{3.35}$$

Put

$$\Delta\theta = \theta(t,p+h) - \theta(t,p), \quad \text{for } t \in [0,T], \text{ and } |h| < \delta. \tag{3.36}$$

It follows from (3.36) and the Continuous Dependence on Initial Conditions Theorem (Theorem 2.4.1) that

$$|h| \leqslant \delta \Rightarrow |\Delta\theta| \leqslant |h|e^{KT}, \quad \text{for all } t \in [0,T], \tag{3.37}$$

for some constant K.

Using (3.34) we can write

$$\begin{aligned} f(\theta(t,p+h)) &= f(\theta(t,p) + \Delta\theta) \\ &= f(\theta(t,p)) + f'(\theta(t,p))\Delta\theta + R(\theta(t,p), \Delta\theta), \end{aligned}$$

so that

$$f(\theta(t,p+h)) - f(\theta(t,p)) \quad = \quad f'(\theta(t,p))\Delta\theta + R(\theta(t,p),\Delta\theta), \tag{3.38}$$

Substituting (3.38) into (3.33) yields

$$g(t,h) = \int_0^t f'(\theta(\tau,p))[\Delta\theta - v(\tau,p)h]\ \mathrm{d}\tau + \int_0^t R(\theta(\tau,p),\Delta\theta)\ \mathrm{d}\tau,$$

or

$$g(t,h) = \int_0^t f'(\theta(\tau,p))g(\tau,p)\ \mathrm{d}\tau + \int_0^t R(\theta(\tau,p),\Delta\theta)\ \mathrm{d}\tau, \tag{3.39}$$

for $t \in [0,T]$ and $|h| < \delta$, where we have used the definition of g in (3.28) and the definition of $\Delta\theta$ in (3.36).

Next, use the assumption that f is C^1 to obtain

$$M = \max_{t\in[0,T]} |f'(\theta(t,p))|. \tag{3.40}$$

Take the absolute value on both sides of (3.39), apply the triangle inequality, and use (3.40) to obtain the estimate

$$|g(t,h)| \leqslant M \int_0^t |g(\tau,p)|\ \mathrm{d}\tau + \int_0^t |R(\theta(\tau,p),\Delta\theta)|\ \mathrm{d}\tau, \tag{3.41}$$

Let $\varepsilon > 0$ be arbitrary. By (3.35), there exists $\eta > 0$ such that

$$|\Delta\theta| < \eta \Rightarrow |R(\theta(t,p),\Delta\theta)| < \frac{\varepsilon}{Te^{(M+K)T}}|\Delta\theta|, \tag{3.42}$$

for all $t \in [0,T]$. Note that, in order to obtain the uniform estimate in (3.42), we have also used the fact that the set

$$C = \{\theta(t,p) \mid t \in [0,T]\}$$

is a compact subset of U.

Next, by making δ smaller, if necessary, so that

$$\delta < \eta e^{-KT}, \tag{3.43}$$

it follows from (3.43), (3.37) and (3.42) that

$$|h| < \delta \Rightarrow |R(\theta(t,p),\Delta\theta)| < \frac{\varepsilon}{T}\ |h|e^{-MT}, \quad \text{for all } t \in [0,T]. \tag{3.44}$$

Thus, combining the estimates in (3.41) and (3.44), we obtain that, if $|h| < \delta$, where δ satisfies (3.43),

$$|g(t,h)| \leqslant \varepsilon|h|e^{-MT} + M\int_0^t |g(\tau,p)|\ \mathrm{d}\tau, \quad \text{for all } t \in [0,T]. \tag{3.45}$$

Applying the result of Problem 5 in Assignment 1 (Gronwall's Inequality) to the estimate in (3.45) we see that

$$|h| < \delta \Rightarrow |g(t,h)| \leqslant \varepsilon |h| e^{-MT} e^{Mt}, \quad \text{for all } t \in [0,T],$$

from which we get that

$$|h| < \delta \Rightarrow |g(t,h)| \leqslant \varepsilon |h|, \quad \text{for all } t \in [0,T].$$

This proves (3.29). Hence,

$$\lim_{h \to 0} \frac{|\theta(t,p+h) - \theta(t,p) - v(t,p)h|}{|h|} = 0,$$

which implies (3.27). Thus, the partial derivative of $\theta(t,p)$ with respect to p exists and

$$\frac{\partial \theta}{\partial p}(t,p) = v(t,p),$$

where $v_p(t) = v(t,p)$ for all $t \in J_p$ solves the linear IVP in (3.26). The IVP in (3.26) can be solved to yield

$$v(t,p) = \exp\left(\int_0^t f'(\theta(\tau,p))\, \mathrm{d}\tau \right). \tag{3.46}$$

Since f' is continuous on U, it follows from (3.46) that $\dfrac{\partial \theta}{\partial p}$ is continuous on $\mathcal{D}$.

Proof of Proposition 3.2.2: First observe that

$$\frac{\partial \theta}{\partial t}(t,p) = u_p'(t)$$

for all $t \in J_p$, where $u_p \colon J_p \to U$ is the C^1 solution to the IVP in (3.13). Thus,

$$\frac{\partial \theta}{\partial t}(t,p) = F(\theta(t,p),$$

and the continuity of $\dfrac{\partial \theta}{\partial t}$ on $\mathcal{D}$ follows from the continuity of the flow map, θ, and the vector field, F.

Next, let $(t,p) \in \mathcal{D}$ and assume that $t > 0$. According to Theorem 2.4.1, for $T > t$, there exists $r = r(T) > 0$ be such that $\theta(t, p+h)$ is defined for all $t \in [0,T]$ and all $h \in \mathbb{R}^N$ such that $\|h\| \leqslant r$. Fix $t \in (0,T)$ and define

$$\theta_t \colon B_r(p) \to U$$

by

$$\theta_t(q) = \theta(t,q), \quad \text{for all } q \in B_r(p).$$

We first prove the map θ_t is differentiable at p; in other words, there exists a linear transformation

$$D\theta_t(p)\colon \mathbb{R}^N \to \mathbb{R}^N$$

such that

$$\theta(t, p+h) = \theta(t,p) + D\theta_t(p)h + R(p,h), \qquad \text{for } h \in \mathbb{R}^N \text{ with } \|h\| \leqslant r, \quad (3.47)$$

where

$$R(p,h) = o(\|h\|), \qquad \text{as } \|h\| \to 0. \quad (3.48)$$

Following the outline of the argument given in Example 3.2.3 for the one–dimensional case, we will show that

$$D\theta_t(p) = V(t,p),$$

where $V(t,p)$ is an $N \times N$ matrix which solves the IVP

$$\begin{cases} \dfrac{dY}{dt} = A(t,p)Y; \\ \\ Y(0) = I, \end{cases} \quad (3.49)$$

where $A(t,p)$ is the matrix given by

$$A(t,p) = DF(\theta(t,p)), \qquad \text{for } t \in J_p,\ p \in U; \quad (3.50)$$

I denotes the $N \times N$ identity matrix, and

$$Y(t) = [y_{ij}(t)]$$

denotes a matrix–valued function of the real parameter t.

The fundamental theory developed in Chapter 2 applies to the IVP in (3.49). In fact, it follows from the definition of $A(t,p)$ in (3.50), the fact that the flow map, θ, is continuous, and the assumption that F is C^1 that A is continuous. Thus, since the equation in (3.50) is linear, we obtain that the IVP in (3.50) has a unique solution, $V(t,p)$, defined on for $t \in J_p$. We show that

$$D\theta_t(p) = V(t,p);$$

in other words, according to (3.47) and (3.48), we show that

$$\lim_{\|h\| \to 0} \frac{\|\theta(t,p+h) - \theta(t,p) - V(t,p)h\|}{\|h\|} = 0. \quad (3.51)$$

As in the one–dimensional situation discussed in Example 3.2.3, we define

$$g(t,h) = \theta(t,p+h) - \theta(t,p) - V(t,p)h, \qquad \text{for } t \in [0,T], \quad \text{and } \|h\| < r. \quad (3.52)$$

Next, as in (3.30)–(3.32) in Example 3.2.3, write

$$\theta(t,p) = p + \int_0^t F(\theta(\tau,p))\ \mathrm{d}\tau, \quad (3.53)$$

$$\theta(t, p+h) = p + h + \int_0^t F(\theta(\tau, p+h))\ \mathrm{d}\tau, \tag{3.54}$$

and

$$V(t,p) = I + \int_0^t DF(\theta(\tau,p))V(\tau,p)\ \mathrm{d}\tau, \tag{3.55}$$

for $t \in [0,T]$ and $\|h\| < r$, where the integral in (3.55) is understood as a matrix integral (i.e., the integral is a matrix whose components are integrals). Substituting (3.53)–(3.55) into (3.52) then yields

$$g(t,h) = \int_0^t [F(\theta(\tau,p+h)) - F(\theta(\tau,p)) - DF(\theta(\tau,p))V(\tau,p)h]\ \mathrm{d}\tau, \tag{3.56}$$

for $t \in [0,T]$ and $\|h\| < r$. Now, using the assumption that F is differentiable on U, we can write

$$F(x + \Delta x) = F(x) + DF(x)\Delta x + R(x, \Delta x), \tag{3.57}$$

where

$$R(x, \Delta x) = o(\|\Delta x\|), \quad \text{as } \|\Delta x\| \to 0. \tag{3.58}$$

Put

$$\Delta\theta = \theta(t, p+h) - \theta(t,p), \quad \text{for } t \in [0,T], \text{ and } \|h\| < r. \tag{3.59}$$

It follows from (3.59) and the Continuous Dependence on Initial Conditions Theorem (Theorem 2.4.1) that

$$\|h\| \leqslant r \Rightarrow |\Delta\theta| \leqslant \|h\|e^{KT}, \quad \text{for all } t \in [0,T], \tag{3.60}$$

and some constant K.

Using (3.57) we can write

$$\begin{aligned} F(\theta(t,p+h)) &= F(\theta(t,p) + \Delta\theta) \\ &= F(\theta(t,p)) + DF(\theta(t,p))\Delta\theta + R(\theta(t,p), \Delta\theta), \end{aligned}$$

so that

$$F(\theta(t,p+h)) - F(\theta(t,p)) = DF(\theta(t,p))\Delta\theta + R(\theta(t,p), \Delta\theta), \tag{3.61}$$

Substituting (3.61) into (3.57) yields

$$g(t,h) = \int_0^t DF(\theta(\tau,p))[\Delta\theta - V(\tau,p)h]\ \mathrm{d}\tau + \int_0^t R(\theta(\tau,p), \Delta\theta)\ \mathrm{d}\tau,$$

or

$$g(t,h) = \int_0^t DF(\theta(\tau,p))g(\tau,p)\ \mathrm{d}\tau + \int_0^t R(\theta(\tau,p), \Delta\theta)\ \mathrm{d}\tau, \tag{3.62}$$

for $t \in [0,T]$ and $\|h\| < r$, where we have used the definition of g in (3.52) and the definition of $\Delta\theta$ in (3.59).

Next, use the assumption that F is C^1 to obtain

$$M = \max_{t\in[0,T]} \|DF(\theta(t,p))\|. \tag{3.63}$$

Take the Euclidean norm on both sides of (3.62), apply the triangle inequality, and use (3.63) to obtain the estimate

$$\|g(t,h)\| \leqslant M\int_0^t \|g(\tau,p)\|\ \mathrm{d}\tau + \int_0^t \|R(\theta(\tau,p),\Delta\theta)\|\ \mathrm{d}\tau, \tag{3.64}$$

Let $\varepsilon > 0$ be arbitrary. By (3.58), there exists $\eta > 0$ such that

$$\|\Delta\theta\| < \eta \Rightarrow \|R(\theta(t,p),\Delta\theta)\| < \frac{\varepsilon}{Te^{(M+K)T}}\|\Delta\theta\|, \tag{3.65}$$

for all $t \in [0,T]$. Note that, in order to obtain the uniform estimate in (3.65), we have also used the fact that the set

$$C = \{\theta(t,p) \mid t \in [0,T]\}$$

is a compact subset of U.

Next, by making r smaller, if necessary, so that

$$r < \eta e^{-KT}, \tag{3.66}$$

it follows from (3.66), (3.65) and (3.60) that

$$\|h\| < r \Rightarrow \|R(\theta(t,p),\Delta\theta)\| < \frac{\varepsilon}{T}\ \|h\|e^{-MT}, \quad \text{for all } t\in[0,T]. \tag{3.67}$$

Thus, combining the estimates in (3.64) and (3.67), we obtain that, if $\|h\| < r$, where r satisfies (3.66),

$$\|g(t,h)\| \leqslant \varepsilon\|h\|e^{-MT} + M\int_0^t \|g(\tau,p)\|\ \mathrm{d}\tau, \quad \text{for all } t\in[0,T]. \tag{3.68}$$

Applying Gronwall's Inequality to the estimate in (3.68) we see that

$$\|h\| < r \Rightarrow \|g(t,h)\| \leqslant \varepsilon\|h\|e^{-MT}e^{Mt}, \quad \text{for all } t\in[0,T],$$

from which we get that

$$\|h\| < r \Rightarrow \|g(t,h)\| \leqslant \varepsilon\|h\|, \quad \text{for all } t\in[0,T].$$

This proves (3.51) by virtue of (3.52); that is,

$$\lim_{\|h\|\to 0} \frac{\|\theta(t,p+h) - \theta(t,p) - V(t,p)h\|}{\|h\|} = 0,$$

which shows that θ_t is differentiable at p with

$$D\theta_t(p) = V(t,p),$$

where $V(\cdot,p)$ is the solution to the IVP (3.49) on J_p.

To complete the proof of Proposition 3.2.2, we first show that $V(t,p)$ depends continuously on the parameter p.

Let T and r be as in the first part of this proof. By virtue of the continuity of the map $t \mapsto V(t,p)$, for fixed $p \in U$, and the assumption that F is C^1, we can find constants M_1 and M_2 with

$$M_1 = \max_{t\in[0,T]} \|V(t,p)\|, \tag{3.69}$$

and

$$M_2 = \max_{\substack{t\in[0,T]\\ \|h\|\leqslant r}} \|DF(\theta(t,p+h)\|. \tag{3.70}$$

Using the integral representation (3.55) for the solution to the IVP (3.49) we obtain, for $\|h\| \leqslant r$, that

$$V(t,p+h) - V(t,p) = \int_0^t [DF(\theta(\tau,p+h))V(\tau,p+h) - DF(\theta(\tau,p))V(\tau,p)]\ \mathrm{d}\tau,$$

which may be written as

$$\begin{aligned} V(t,p+h) - V(t,p) \quad &= \quad \int_0^t DF(\theta(\tau,p+h))[V(\tau,p+h) - V(\tau,p)]\ \mathrm{d}\tau \\ &\quad + \int_0^t [DF(\theta(\tau,p+h)) - DF(\theta(\tau,p))]V(\tau,p)\ \mathrm{d}\tau. \end{aligned}$$

Taking Euclidean norms on both sides of the pervious equation, applying the triangle inequality, and using (3.69) and (3.70), we obtain that

$$\begin{aligned} \|V(t,p+h) - V(t,p)\| \quad &\leqslant \quad M_2 \int_0^t \|V(\tau,p+h) - V(\tau,p)\|\ \mathrm{d}\tau \\ &\quad + M_1 \int_0^t \|DF(\theta(\tau,p+h)) - DF(\theta(\tau,p))\|\ \mathrm{d}\tau. \end{aligned} \tag{3.71}$$

Let $\varepsilon > 0$ be given. Using the continuity of DF and the of the flow map θ, we obtain $\delta > 0$ such that $\delta < r$ and

$$\|h\| < \delta \Rightarrow \|\|DF(\theta(t,p+h)) - DF(\theta(t,p))\| < \frac{\varepsilon}{2TM_1e^{M_2T}}, \tag{3.72}$$

for all $t \in [0,T]$. It then follows from (3.71) and (3.72) that $\|h\| < \delta$ implies that

$$\|V(t,p+h) - V(t,p)\| < M_2 \int_0^t \|V(\tau,p+h) - V(\tau,p)\|\ \mathrm{d}\tau + \frac{\varepsilon}{2e^{M_2T}}, \tag{3.73}$$

for all $t \in [0, T$. Applying Gronwall's inequality to (3.73) then yields that

$$\|V(t, p+h) - V(t, p)\| < \frac{\varepsilon}{2e^{M_2 T}} e^{M_2 t}, \quad \text{for all } t \in [0, T], \tag{3.74}$$

provided that $\|h\| < \delta$. It follows from (3.74) that

$$\|h\| < \delta \Rightarrow \|V(t, p+h) - V(t, p)\| < \frac{\varepsilon}{2}, \quad \text{for all } t \in [0, T]. \tag{3.75}$$

It remains to show that the map

$$(t, p) \mapsto V(t, p)$$

is continuous for $t \in [0, T]$ and $p \in U$. Let $\varepsilon > 0$ be given and let $s \in \mathbb{R}$ be such that $t + s \in [0, T]$. Choose $\delta > 0$ as in the previous part of this proof so that (3.75) holds true. By the continuity of the map

$$t \mapsto V(t, p), \text{ for } t \in J_p,$$

way may assume, by making δ smaller if necessary that

$$|s| < \delta \Rightarrow t + s \in [0, T] \text{ and } \|V(t+s, p) - V(t, p)\| < \frac{\varepsilon}{2}. \tag{3.76}$$

It then follows from (3.75) and (3.76) that $|s| < \delta$ and $\|h\| < \delta$ implies that

$$\begin{aligned} \|V(t+s, p+h) - V(t, p)\| &= \|(V(t+s, p+h) - V(t+s, p)) + (V(t+s, p) - V(t, p))\| \\ &\leqslant \|V(t+s, p+h) - V(t+s, p)\| + \|V(t+s, p) - V(t, p)\| \\ &< \varepsilon, \end{aligned}$$

and the proof of Proposition 3.2.2 is now complete. ■

Chapter 4

Continuous Dynamical Systems

We saw in the previous chapter that for any C^1 vector field, $F\colon U \to \mathbb{R}^N$, defined in an open set $U \subseteq \mathbb{R}^N$, there exists a corresponding flow map, $\theta\colon \mathcal{D} \to U$, defined on a flow domain $\mathcal{D} \subseteq \mathbb{R} \times U$. In other words, for each $p \in U$, the function $u_p\colon J_p \to U$ defined by

$$u_p(t) = \theta(t, p), \quad \text{for all } t \in J_p,$$

is the unique solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p, \end{cases} \tag{4.1}$$

defined on a maximal interval of existence, J_p. We also saw in Chapter 3 that the flow map, $\theta\colon \mathcal{D} \to U$, is a C^1 map. In this chapter we begin by noting that, in addition to being C^1, the flow map, θ, satisfies

$$\theta(0, p) = p, \quad \text{for all } p \in U, \tag{4.2}$$

and

$$\theta(t + s, p) = \theta(t, \theta(s, p)), \tag{4.3}$$

for all $p \in U$, and all $t, s \in \mathbb{R}$ with $t + s, s \in J_p$ and $t \in J_{\theta(s,p)}$. The identity in (4.2) follows from the fact that the map $t \mapsto \theta(t, p)$, solves the IVP (4.1). The identity in (4.3) was proved in Problem 1 of Assignment #2.

For the case in which the identity in (4.3) holds true for all $t, s \in \mathbb{R}$, the flow of F defines an action of the group $\mathbb{R}$, under addition, on the set U. In general if G denotes a group, and S a set, an action of G on S is a map

$$\varphi\colon G \times S \to S$$

satisfying

$$\varphi(e,s) = s, \quad \text{for all } s \in S, \tag{4.4}$$

where e is the group identity, and

$$\varphi(gh,s) = \varphi(g,\varphi(h,s)) \quad \text{for all } s \in S, \text{ and all } g,h \in G. \tag{4.5}$$

In the case in which $\theta(t,p)$ is defined for all $t \in \mathbb{R}$ and $p \in U$, we see that $\varphi(t,p) = \theta(t,p)$ for all $(t,p) \in \mathbb{R} \times U$ satisfies the group action axioms in (4.4) and (4.5) for $G = \mathbb{R}$, with addition as the group operation, and $S = U$, in view of (4.2) and (4.3), respectively. For the particular case in which the action of $\mathbb{R}$ on U is defined by the flow map of a C^1 vector field on U, the map $(t,p) \mapsto \theta(t,p)$ is C^1; this will be the definition of a continuous dynamical system that we will use in these notes.

4.1 Definition of Continuous Dynamical Systems

A continuous dynamical system on an open set $U \subseteq \mathbb{R}^N$ is a C^1 map

$$\theta \colon \mathbb{R} \times U \to U$$

which satisfies the group action axioms:

$$\theta(0,p) = p, \quad \text{for all } p \in U, \tag{4.6}$$

and

$$\theta(t+s,p) = \theta(t,\theta(s,p)), \quad \text{for all } p \in U, \quad \text{and all } t,s \in \mathbb{R}. \tag{4.7}$$

Thus, a continuous dynamical system can be thought of as a C^1 action of the group $(\mathbb{R},+)$ on the open set $U \subseteq \mathbb{R}^N$.

Example 4.1.1. Let $F \colon U \to \mathbb{R}^N$ be a C^1 vector field. Then, the flow map, θ, of F defined in the previous chapter is a continuous dynamical system if, for all $p \in U$, the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p, \end{cases}$$

has a solution that exists for all $t \in \mathbb{R}$.

Example 4.1.2. Define $\theta \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \text{ for } t,p,q \in \mathbb{R}.$$

To see that θ defines a dynamical system, first observe that

$$\theta\left(0, \begin{pmatrix} p \\ q \end{pmatrix}\right) = I \begin{pmatrix} p \\ q \end{pmatrix},$$

where I denotes the 2×2 identity matrix. Consequently,

$$\theta\left(0, \begin{pmatrix} p \\ q \end{pmatrix}\right) = \begin{pmatrix} p \\ q \end{pmatrix}.$$

Finally, use the trigonometric identities

$$\begin{aligned} \cos(t+s) &= \cos t \cos s - \sin t \sin s \\ \sin(t+s) &= \cos t \sin s + \sin t \cos s \end{aligned}$$

to verify that

$$\theta\left(t+s, \begin{pmatrix} p \\ q \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

from which we obtain that

$$\theta\left(t+s, \begin{pmatrix} p \\ q \end{pmatrix}\right) = \theta\left(t, \theta\left(s, \begin{pmatrix} p \\ q \end{pmatrix}\right)\right).$$

For each $t \in \mathbb{R}$, the dynamical system, $\theta\colon \mathbb{R} \times U \to U$, induces a map on U, denoted by $\theta_t\colon U \to U$, and given by

$$\theta_t(p) = \theta(t, p), \qquad \text{for all } p \in U. \tag{4.8}$$

The map $\theta_t\colon U \to U$ defined by (4.8) is C^1. Furthermore, it follows from (4.6) and (4.7) that

$$\theta_t \circ \theta_{-t} = id,$$

where id denotes the identity map in U, and

$$\theta_{-t} \circ \theta_t = id.$$

It then follows that θ_t is invertible with inverse θ_{-t}. Hence, θ_t is C^1 with an inverse which is also C^1. We say that θ_t is a diffeomorphism of U. Hence, a dynamical system θ_t induces a family of diffeomorphisms, $\{\theta_t\}_{t\in\mathbb{R}}$, of the set U into itself.

Example 4.1.3. For the dynamical system defined in Example 4.1.2,

$$\theta_t\colon \mathbb{R}^2 \to \mathbb{R}^2$$

is a rotation of the plane through an angle of t (in radians) of the plane $\mathbb{R}^2$.

4.2 Orbits

In the study of group actions, it is of interest to look at orbits of points in the set on which the group acts.

Definition 4.2.1 (Orbits)**.** Let $\theta(t,p)$ denote a dynamical system on an open set $U \subseteq \mathbb{R}^N$. Given $p \in U$, the orbit of the flow, θ_t, through p is the set, γ_p, defined by

$$\gamma_p = \{x \in U \mid x = \theta(t,p) \text{ for some } t \in \mathbb{R}\};$$

in other words, γ_p is the image of the map $\theta_p \colon \mathbb{R} \to U$ defined by

$$\theta_p(t) = \theta(t,p), \qquad \text{for all } t \in \mathbb{R}.$$

Example 4.2.2. For the dynamical system $\theta \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given in Example 4.1.2, $\gamma_{(p,q)}$ is a circle of radius $\sqrt{p^2+q^2}$ around the origin for the case $(p,q) \neq (0,0)$; and $\gamma_{(0,0)} = \{(0,0)\}$. Figure 4.2.1 shows $\gamma_{(1,0)}$, $\gamma_{(0,0)}$ and another typical

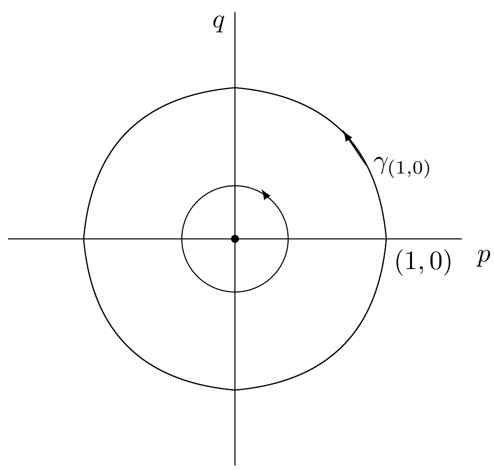

Figure 4.2.1: Phase Portrait of θ in Example 4.1.2

orbit of the dynamical system θ defined in Example 4.1.2. The arrows on the the two circular orbits portrayed in the figure indicate the direction on the orbit induced by the parametrization $\theta_{(p,q)} \colon \mathbb{R} \to \mathbb{R}^2$, defined by

$$\theta_{(p,q)}(t) = \theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right), \qquad \text{for all } t \in \mathbb{R}, \tag{4.9}$$

at t increases.

Definition 4.2.3 (Phase Portrait)**.** A depiction of all possible kinds of orbits that a dynamical system can have is known as a phase portrait of the system.

Example 4.2.4. Figure 4.2.1 shows the phase portrait of the dynamical system, $\theta \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, given in Example 4.1.2. Observe that, for $(p,q) \neq (0,0)$ and

$\theta_{(p,q)}$ as defined in (4.9),

$$\begin{aligned}\|\theta_{(p,q)}(t)\|^2 &= (p\cos t - q\sin t)^2 + (p\sin t + q\cos t)^2\\ &= p^2\cos^2 t - 2pq\sin t\cos t + q^2\sin^2 t\\ &\qquad +p^2\sin^2 t + 2pq\sin t\cos t + q^2\cos^2 t\\ &= p^2+q^2, \quad \text{for all } t\in\mathbb{R},\end{aligned}$$

which shows that $\gamma_{(p,q)}$ lies in the circle of radius $r = \sqrt{p^2+q^2}$ around the origin in $\mathbb{R}^2$. On the other hand, if $(x,y) \in S_r((0,0))$, the circle of radius r around the origin in $\mathbb{R}^2$, where $r > 0$, by letting

$$t = \arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{q}{p}\right),$$

we can show that

$$\theta_{(p,q)}(t) = \begin{pmatrix} x\\ y\end{pmatrix}.$$

In other words, $(x,y) \in \gamma_{(p,q)}$. Consequently, $\gamma_{(p,q)} = S_r((0,0))$, for $r = \sqrt{p^2+q^2}$, as claimed in Example 4.2.2. On the other hand, if $(p,q) = (0,0)$, then $\gamma_{(p,q)} = \{(0,0)\}$. Thus, the singleton $\{(0,0)\}$ and concentric circles around the origin are the only kinds of orbits that the dynamical system, $\theta(t,p)$, defined in Example 4.1.2 can have.

4.3 Infinitesimal Generator of a Dynamical System

Given a dynamical system, $\theta\colon \mathbb{R}\times U \to U$, on an open set $U \subseteq \mathbb{R}^N$, we can define a vector field, $F\colon U \to \mathbb{R}^N$, as follows

$$F(x) = \frac{\partial}{\partial t}\theta(t,x)\Big|_{t=0}, \quad \text{for all } x \in U; \tag{4.10}$$

in other words,

$$F(x) = \lim_{h\to 0}\frac{\theta(h,x)-\theta(0,x)}{h}, \quad \text{for all } x\in U. \tag{4.11}$$

Since we are assuming that the dynamical system, $\theta\colon \mathbb{R}\times U \to U$ is a C^1 map, it follows that $F\colon U\to\mathbb{R}^N$ defined in (4.10) is a C^1 vector field defined in U. We show next that $\theta\colon \mathbb{R}\times U\to U$ is the flow map for the vector field F. The vector field, F, defined in (4.11) is called the infinitesimal generator of the dynamical system θ_t, for $t\in\mathbb{R}$.

Thus, we need to show that the map, $\theta_p\colon \mathbb{R}\to U$, given by

$$\theta_p(t) = \theta(t,p), \quad \text{for all } t\in\mathbb{R} \tag{4.12}$$

is the unique solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p, \end{cases} \tag{4.13}$$

Using the group action axiom for $\theta(t,p)$ in (4.7), we see that θ_p defined in (4.12) satisfies

$$\theta_p(t+h) = \theta(t+h,p) = \theta(h,\theta(t,p)), \quad \text{for } t,h \in \mathbb{R}.$$

We then have that

$$\theta_p(t+h) = \theta(h,\theta_p(t)), \quad \text{for } t,h \in \mathbb{R}. \tag{4.14}$$

Thus, for $h \neq 0$ we obtain from (4.14) that

$$\frac{\theta_p(t+h) - \theta_p(t)}{h} = \frac{\theta(h,\theta_p(t)) - \theta(0,\theta_p(t))}{h}, \tag{4.15}$$

where we have also used the group action axiom for $\theta(t,p)$ in (4.6).

Next, letting $h \to 0$ in (4.15) and using the definition of the field, F, in (4.11), we obtain that

$$\theta_p'(t) = F(\theta_p(t)),$$

which shows that θ_p solves the differential equation in the IVP (4.13). Finally, since $\theta_p(0) = p$, by the group action axiom for $\theta(t,p)$ in (4.6), we see that $\theta_p \colon \mathbb{R} \to U$ solves the IVP in (4.13), which was to be shown.

Remark 4.3.1. We have seen that every continuous dynamical system, $\theta \colon \mathbb{R} \times U \to U$, has an associated C^1 vector field, $F \colon U \to \mathbb{R}^N$, given by (4.10); namely, the infinitesimal generator of θ. However, it is not the case that every C^1 vector field, F, has a dynamical system, $\theta \colon \mathbb{R} \times U \to U$, in the sense defined in Section 4.1. The issue at hand is that solutions to the IVP (4.13), for given $p \in U$, might not be defined for all $t \in \mathbb{R}$.

Example 4.3.2. Let $\theta \colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ be the dynamical system given in Example 4.1.2. To find the infinitesimal generator of θ, we first compute

$$\frac{\partial\theta}{\partial t}\left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2;$$

so that

$$\begin{aligned} F\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{\partial\theta}{\partial t}\left(0, \begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} -y \\ x \end{pmatrix}, \end{aligned}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. It then follows that the infinitesimal generator of θ is the vector field, $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

In other words, the dynamical system, $\theta\colon \mathbb{R} \times \mathbb{R}^2$, given by

$$\theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \text{ for } t, p, q \in \mathbb{R},$$

is the flow of the linear system of differential equations

$$\begin{cases} \dfrac{dx}{dt} & = & -y; \\[2ex] \dfrac{dy}{dt} & = & x. \end{cases}$$

4.4 Fixed Points and Equilibrium Solutions

Let $\theta\colon \mathbb{R} \times U \to U$ be a C^1 dynamical system in U with infinitesimal generator $F\colon U \to \mathbb{R}^N$. A fixed point of the flow θ_t in U is a point $p^* \in U$ such that

$$\theta(t, p^*) = p^*, \quad \text{for all } t \in \mathbb{R},$$

or

$$\theta_{p^*}(t) = p^*, \quad \text{for all } t \in \mathbb{R}. \tag{4.16}$$

Taking the derivative with respect to t on both sided of (4.16) we obtain that

$$\theta'_{p^*}(t) = 0, \quad \text{for all } t \in \mathbb{R},$$

so that, by the definition of the infinitesimal generator of θ_t,

$$F(\theta_{p^*}(t)) = 0, \quad \text{for all } t \in \mathbb{R}. \tag{4.17}$$

Combining (4.17) with (4.16) we obtain

$$F(p^*) = 0. \tag{4.18}$$

Thus, a fixed point of the dynamical system with infinitesimal generator $F\colon U \to \mathbb{R}^N$ is a solution to the equation

$$F(p) = 0. \tag{4.19}$$

Solutions to (4.19) are also known as equilibrium points, or singular points. A point $p \in U$ which is not a fixed point of the system generated by F is called a regular point of F. A solution to the system

$$\frac{dx}{dt} = F(x),$$

defined by (4.16) where $p^* \in U$ satisfies (4.18) is called an equilibrium solution.

Example 4.4.1. Consider the two–dimensional system

$$\begin{cases} \dfrac{dx}{dt} = \lambda x + y; \\ \\ \dfrac{dy}{dt} = \lambda y, \end{cases} \tag{4.20}$$

where $\lambda \neq 0$.

In this case the field, $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, is given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x + y \\ \lambda y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

The equilibrium points are therefore solutions to the system

$$\begin{cases} \lambda x + y = 0; \\ \lambda y = 0. \end{cases} \tag{4.21}$$

Since $\lambda \neq 0$, the only solution to the system in (4.21) is the origin, $(0, 0)$, in $\mathbb{R}^2$. Thus, $(0, 0)$ is the only equilibrium point of the system in (4.20).

4.5 Limit Sets

The dynamical system corresponding to the two–dimensional system (4.20) in Example 4.4.1 is given by

$$\theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \quad \text{for } t \in \mathbb{R}, \ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^2. \tag{4.22}$$

Thus, if $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in \gamma_{\binom{p}{q}}$, the orbit of $\begin{pmatrix} p \\ q \end{pmatrix}$, it follows from (4.22) that

$$\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| \leqslant e^{\lambda t}\sqrt{2 + t^2}\left\| \begin{pmatrix} p \\ q \end{pmatrix} \right\|, \quad \text{for all } t \in \mathbb{R}. \tag{4.23}$$

Hence, if $\lambda < 0$, we obtain from (4.23) that

$$\lim_{t \to \infty} \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| = 0,$$

from which we conclude that, if $\lambda < 0$, then

$$\lim_{t \to \infty} \theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We then say that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an ω–limit point ("omega limit point") of the orbit $\gamma_{\binom{p}{q}}$.

Definition 4.5.1 (Limit Sets)**.** Let $\theta\colon \mathbb{R}\times U \to U$ denote a dynamical system on an open set $U \subseteq \mathbb{R}^N$, and let γ_p, for $p \in U$, be an the orbit of the point p under the flow θ_t.

- *(ω–limit point)* A point $q \in U$ is said to be an ω–limit point of γ_p if there exists a sequence of real values, (t_m), such that $t_m \to +\infty$ as $m \to \infty$ and
$$\lim_{m\to\infty} \theta(t_m, p) = q.$$
- *(ω–limit set)* The set of all ω–limit sets of the orbit γ_p is called the ω–limit set of γ_p and is denoted by $\omega(\gamma_p)$.
- *(α–limit point)* A point $q \in U$ is said to be an α–limit point of γ_p if there exists a sequence of real values, (t_m), such that $t_m \to -\infty$ as $m \to \infty$ and
$$\lim_{m\to\infty} \theta(t_m, p) = q.$$
- *(α–limit set)* The set of all α–limit sets of the orbit γ_p is called the α–limit set of γ_p and is denoted by $\alpha(\gamma_p)$.

Example 4.5.2. Let $\theta\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ be the dynamical system given in (4.22) corresponding to the two–dimensional system in (4.20). If $\lambda < 0$, then
$$\omega(\gamma_{\binom{p}{q}}) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\},$$
for all $\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^2$.

On the other hand, if $\begin{pmatrix} p \\ q \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $\lambda < 0$, then
$$\alpha(\gamma_{\binom{p}{q}}) = \emptyset.$$
To see why the last claim is true, note that, if $q \neq 0$,
$$\left\|\theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right)\right\| \geqslant |q|e^{\lambda t} \to \infty, \qquad \text{as } t \to -\infty.$$
Consequently,
$$\left\|\theta\left(t_m, \begin{pmatrix} p \\ q \end{pmatrix}\right)\right\| \to \infty, \qquad \text{as } m \to \infty,$$
for any sequence of real numbers, (t_m), such that $t_m \to -\infty$ as $m \to \infty$; therefore,
$$\lim_{m\to\infty} \theta\left(t_m, \begin{pmatrix} p \\ q \end{pmatrix}\right)$$
does not exist for the case $q \neq 0$ and $\lambda < 0$.

Next, if $q = 0$ and $p \neq 0$, while $\lambda < 0$, note that

$$\left\| \theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) \right\| \geqslant |p|e^{\lambda t} \to \infty, \quad \text{as } t \to -\infty,$$

and the same result is obtained.

Finally, observe that, regardless of the sign of λ,

$$\alpha(\gamma_{\binom{0}{0}}) = \omega(\gamma_{\binom{0}{0}}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

since $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the equilibrium point of the system.

4.6 Properties of Limit Sets

Definition 4.6.1 (Invariant Sets)**.** Let $\theta\colon \mathbb{R} \times U \to U$ be a dynamical system on U, an open subset of $\mathbb{R}^N$. A subset A of U is said to be invariant under the flow θ if for every $q \in A$, $\theta(t, q) \in A$ for all $t \in \mathbb{R}$.

Example 4.6.2. For $p \in U$, the orbit γ_p is invariant under the flow. In fact, if $q \in \gamma_p$, then $q = \theta(t_o, p)$, for some $t_o \in \mathbb{R}$. We then have that, for any $t \in \mathbb{R}$,

$$\theta(t, q) = \theta(t, \theta(t_o, p)) = \theta(t + t_o, p) \in \gamma_p.$$

In the next proposition we will see that the limit sets of an orbit is also invariant.

Proposition 4.6.3. Let $\theta\colon \mathbb{R} \times U \to U$ be a dynamical system on U, an open subset of $\mathbb{R}^N$. For any $p \in U$, the ω–limit set and α–limit set of γ_p are closed, invariant subsets of U.

Proof: We prove the assertions in the proposition for $\omega(\gamma_p)$; the arguments for $\alpha(\gamma_p)$ are analogous.

We first show that $\omega(\gamma_p)$ is invariant. Let $q \in \omega(\gamma_p)$; then there exists a sequence of real numbers, (t_m), such that $t_m \to \infty$ as $m \to \infty$ and

$$q = \lim_{m\to\infty} \theta(t_m, p). \tag{4.24}$$

Now, by the continuity of the map $\theta_t\colon U \to U$ for each $t \in \mathbb{R}$, we obtain from (4.24) that

$$\theta(t, q) = \lim_{m\to\infty} \theta(t, \theta(t_m, p)) \tag{4.25}$$

for any $t \in \mathbb{R}$. On the other hand,

$$\theta(t, \theta(t_m, p)) = \theta(t + t_m, p) = \theta(t_m, \theta(t, p)). \tag{4.26}$$

Thus, combining (4.25) and (4.26), we see that

$$\theta(t, q) = \lim_{m\to\infty} \theta(t_m, \theta(t, p)),$$

which shows that

$$\theta(t,q) \in \omega(\gamma_{\theta(t,p)}), \quad \text{for any } t \in \mathbb{R}. \tag{4.27}$$

However, $\gamma_{\theta(t,p)} = \gamma_p$, for any $t \in \mathbb{R}$, since $\theta(t,p) \in \gamma_p$, for any $t \in \mathbb{R}$ (see, for instance, Problem 3 in Assignment #4). Thus, (4.27) can be written as

$$\theta(t,q) \in \omega(\gamma_p), \quad \text{for any } t \in \mathbb{R},$$

which shows that $\omega(\gamma_p)$ is invariant under the flow.

Next, we show that $\omega(\gamma_p)$ is closed. This is equivalent to showing that the complement, $\omega(\gamma_p)^c$, is open.

Arguing by contradiction, suppose that there exist $q \in \omega(\gamma_p)^c$ and a sequence of points, (q_m), in $\omega(\gamma_p)$ such that

$$\lim_{m\to\infty} \|q_m - q\| = 0. \tag{4.28}$$

Now, since each q_m is in $\omega(\gamma_p)$, we can construct a sequence of real numbers, (t_m) such that $t_m \to \infty$ as $m \to \infty$, and

$$\|\theta(t_m,p) - q_m\| < \frac{1}{m}, \quad \text{for all } m \in \mathbf{N}. \tag{4.29}$$

Applying the triangle inequality we obtain that

$$\|\theta(t_m,p) - q\| < \frac{1}{m} + \|q_m - q\|, \quad \text{for all } m \in \mathbf{N}, \tag{4.30}$$

where we have also used the estimate in (4.29). It follows from (4.30), (4.29) and the Squeeze Lemma that

$$\lim_{m\to\infty} \|\theta(t_m,p) - q\| = 0,$$

which shows that $q \in \omega(\gamma_p)$, contradicting the assumption that $q \in \omega(\gamma_p)^c$. This contradiction shows that $\omega(\gamma_p)^c$ is open, and therefore $\omega(\gamma_p)$ is closed. ■

It is possible for the ω–limit set of an orbit to be empty. For instance, let $\theta\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\theta\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{for } t \in \mathbb{R},\ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^2, \tag{4.31}$$

where $\lambda > 0$, and consider the orbit $\gamma_{\binom{1}{0}}$. Points, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, in $\gamma_{\binom{1}{0}}$ are of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}.$$

We then have that

$$\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| = e^{\lambda t} \to \infty, \text{ as } t \to \infty,$$

since $\lambda > 0$. It then follows that $\omega(\gamma_{\binom{1}{0}}) = \emptyset$.

The reason the limit of $\theta\left(t, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ as $t \to \infty$ fails to exist is that the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is unbounded for positive values of t.

Definition 4.6.4 (Semi–Orbits)**.** Let $\theta\colon \mathbb{R} \times U \to U$ denote a dynamical system on an open set $U \subseteq \mathbb{R}^N$. Given $p \in U$, the forward semi–orbit of the flow, θ_t, through p is the set, γ_p^+, defined by

$$\gamma_p^+ = \{x \in U \mid x = \theta(t, p) \text{ for some } t \geqslant 0\}.$$

Similarly, the backward semi–orbit through p, denoted by γ_p^-, is the set

$$\gamma_p^- = \{x \in U \mid x = \theta(t, p) \text{ for some } t < 0\}.$$

Observe that $\gamma_p = \gamma_p^- \cup \gamma_p^+$.

Example 4.6.5. Let $\theta\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be as given in (4.31), where $\lambda > 0$; then

$$\gamma^+_{\binom{1}{0}} = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 1,\ y = 0\}$$

and

$$\gamma^-_{\binom{1}{0}} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ y = 0\}.$$

In the next proposition we will see that, if γ_p^+ is bounded, then $\omega(\gamma_p)$ is nonempty. Similarly, if γ_p^- is bounded, then $\alpha(\gamma_p)$ is nonempty. In the previous example, note that $\gamma^-_{\binom{1}{0}}$ is bounded and $\alpha(\gamma_{\binom{1}{0}}) = \{(0, 0)\}$; so that, $\alpha(\gamma_{\binom{1}{0}}) \neq \emptyset$.

Proposition 4.6.6. Let $\theta\colon \mathbb{R} \times U \to U$ be a dynamical system on U, an open subset of $\mathbb{R}^N$, and $p \in U$. If γ_p^+ lies in a compact subset of U, then $\omega(\gamma_p)$ is nonempty, compact and connected. Similarly, if γ_p^- lies in a compact subset of U, then $\alpha(\gamma_p)$ is nonempty, compact and connected.

Proof: We prove the result for the ω–limit set; the arguments for the α–limit set are analogous.

Assume that $\gamma_p^+ \subset K$, where K is a compact subset of U. For any sequence, (t_m), of real numbers such that $t_m \to \infty$ as $m \to \infty$, the set $\{\theta(t_m, p)\}$ is a subset of K. Hence, since K is compact, there exists $q \in K$ and a subsequence, (t_{m_k}) such that $t_{m_k} \to \infty$ as $k \to \infty$, and

$$q = \lim_{k \to \infty} \theta(t_{m_k}, p);$$

that is, $q \in \omega(\gamma_p)$, and therefore $\omega(\gamma_p) \neq \emptyset$.

Next, observe that, if $q \in \omega(\gamma_p)$, then

$$q = \lim_{m\to\infty} \theta(t_m, p),$$

for a sequence of positive numbers, t_m, tending to infinity as $m \to \infty$. We then have that $\theta(t_m, p) \in K$ for all m. Consequently, since K is closed, $q \in K$. Thus,

$$\omega(\gamma_p) \subseteq K.$$

since $\omega(\gamma_p)$ is closed, by Proposition 4.6.6, and K is compact, it follows that $\omega(\gamma_p)$ is compact.

It remains to show that $\omega(\gamma_p)$ is connected. Assume to the contrary that there exist nonempty open subsets, U_1 and U_2, of U such that

$$U_1 \cap U_2 = \emptyset, \tag{4.32}$$

$$U_1 \cap \omega(\gamma_p) \neq \emptyset \quad \text{and} \quad U_2 \cap \omega(\gamma_p) \neq \emptyset, \tag{4.33}$$

and

$$\omega(\gamma_p) \subseteq U_1 \cup U_2. \tag{4.34}$$

Define

$$C_1 = \omega(\gamma_p) \cap U_1 \text{ and } C_1 = \omega(\gamma_p) \cap U_2. \tag{4.35}$$

It follows from (4.32) and (4.34) that

$$C_1 \cap C_2 = \emptyset, \tag{4.36}$$

and

$$\omega(\gamma_p) = C_1 \cup C_2. \tag{4.37}$$

Next, we see that C_1 and C_2 are compact. This follows from the fact that C_1 and C_2 are both closed subsets of $\omega(\gamma_p)$, which was shown to be compact previously in this proof. To see that C_1 is closed, let (q_m) be a sequence of points in C_1 such that

$$\lim_{m\to\infty} \|q_m - q\| = 0, \tag{4.38}$$

for some $q \in \omega(\gamma_p)$. We show that $q \in C_1$. If this is not the case, it follows from (4.37) that $q \in C_2$. Then, by the definition of C_2 in (4.35), $q \in U_2$. It then follows from (4.38) that there exists $M \in \mathbf{N}$ such that

$$m \geqslant M \Rightarrow q_m \in U_2.$$

However, this is impossible in view of (4.32) since $q_m \in U_1$ for all $m \in \mathbf{N}$. We therefore conclude that $q \in C_1$, and therefore C_1 is closed. A similar argument shows that C_2 is closed.

From (4.36) and the fact that C_1 and C_2 are compact, it follows that

$$\delta = \text{dist}(C_1, C_2) > 0. \tag{4.39}$$

By virtue of (4.33) and (4.33), we can find $q_1 \in C_1$ and $q_2 \in C_2$. Then, there exist sequences of real numbers, (t_m) and (s_m), such that

$$t_m < s_m, \quad \text{for all } m \in \mathbf{N}, \tag{4.40}$$

and

$$t_m \to \infty \quad \text{and} \quad s_m \to \infty, \quad \text{as } m \to \infty, \tag{4.41}$$

with

$$\|\theta(t_m, p) - q_1\| < \frac{\delta}{4} \quad \text{and} \quad \|\theta(s_m, p) - q_2\| < \frac{\delta}{4}, \quad \text{for all } m \in \mathbf{N}. \tag{4.42}$$

Hence, for each $m \in \mathbf{N}$,

$$\inf_{q\in C_1} \|\theta(t_m, p) - q\| < \frac{\delta}{4} \quad \text{and} \quad \inf_{q\in C_2} \|\theta(s_m, p) - q\| < \frac{\delta}{4}. \tag{4.43}$$

Define the functions $f_1 \colon \mathbb{R} \to \mathbb{R}$ and $f_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_1(t) = \text{dist}(\theta(t, p), C_1) = \inf_{q\in C_1} \|\theta(t, p) - q\| \tag{4.44}$$

and

$$f_2(t) = \text{dist}(\theta(t, p), C_2) = \inf_{q\in C_2} \|\theta(t, p) - q\| \tag{4.45}$$

for all $t \in \mathbb{R}$. By the continuity of the map $t \mapsto \theta(t, p)$ and the continuity of the distance fucntion, it follows that f_1 and f_2 are continuous functions.

It follows from (4.43) and the definition of f_1 and f_2 in (4.44) and (4.45), respectively, that

$$f_1(t_m) < \frac{\delta}{4} \quad \text{and} \quad f_2(s_m) < \frac{\delta}{4}. \tag{4.46}$$

Next, let $\overline{q}_2 \in C_2$ be such that $f_2(t_m) = \|\theta(t_m, p) - \overline{q}_2\|$. It then follows from (4.39) and the triangle inequality that

$$\delta \leqslant \|q_1 - \overline{q}_2\| \leqslant \|q_1 - \theta(t_m, p)\| + \|\theta(t_m, p) - \overline{q}_2\|,$$

so that

$$\delta < \frac{\delta}{4} + f_2(t_m),$$

where we have used the first inequality in (4.42). We therefore obtain that

$$f_2(t_m) > \frac{3\delta}{4}. \tag{4.47}$$

Combining (4.47) and the first inequality in (4.46) that

$$f_1(t_m) - f_2(t_m) < -\frac{\delta}{2} < 0. \tag{4.48}$$

Similar calculations show that

$$f_1(s_m) - f_2(s_m) > \frac{\delta}{2} > 0. \tag{4.49}$$

It follows from (4.40), (4.48), (4.49), the continuity of f_1 and f_2, and the intermediate value theorem that there exists a sequence, (τ_m), of real numbers with

$$t_m < \tau_m < s_m, \quad \text{for all } m, \tag{4.50}$$

and

$$f_1(\tau_m) - f_2(\tau_m) = 0, \quad \text{for all } m \in \mathbf{N},$$

or

$$\text{dist}(\theta(\tau_m, p), C_1) = \text{dist}(\theta(\tau_m, p), C_2) \quad \text{for all } m \in \mathbf{N}. \tag{4.51}$$

In view of (4.41) and (4.50) we also see that

$$\tau_m \to \infty, \quad \text{as } m \to \infty. \tag{4.52}$$

Since $\theta(\tau_m, p) \in K$, for all m, by assumption, there exists a subsequence, (τ_{m_k}), such that

$$\lim_{k\to\infty} \theta(\tau_{m_k}, p) = \overline{q}, \tag{4.53}$$

for some $\overline{q} \in K$.

It follows from (4.52) and (4.53) that

$$\overline{q} \in \omega(\gamma_p). \tag{4.54}$$

Also, from (4.53), (4.51) and the continuity of the distance function we obtain that

$$\text{dist}(\overline{q}, C_1) = \text{dist}(\overline{q}, C_2). \tag{4.55}$$

From (4.54) and (4.37) we obtain that either $\overline{q} \in C_1$ or $\overline{q} \in C_2$. If $\overline{q} \in C_1$, then

$$\text{dist}(\overline{q}, C_1) = 0, \tag{4.56}$$

and, by virtue of (4.39),

$$\text{dist}(\overline{q}, C_2) \geqslant \delta > 0. \tag{4.57}$$

Observe that (4.56) and (4.57) are in contradiction with (4.55). Thus, we must have that $\overline{q} \in C_2$, which leads to

$$\text{dist}(\overline{q}, C_1) \geqslant \delta > 0 \quad \text{and} \quad \text{dist}(\overline{q}, C_2) = 0,$$

which, again, contradict (4.55). This contradiction establishes that $\omega(\gamma_p)$ is connected, and therefore the proof of Proposition 4.6.6 is now complete. ■

Example 4.6.7. In Example 4.2.4 we showed that the orbits of the system

$$\begin{cases} \dfrac{dx}{dt} = -y; \\[2ex] \dfrac{dy}{dt} = x, \end{cases} \tag{4.58}$$

consist of the singleton $\{(0,0)\}$ and concentric circles around the origin. Some of this orbits are shown in Figure 4.2.1 on page 50. For instance,

$$\gamma_{\binom{1}{0}} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}.$$

We show that

$$\omega(\gamma_{\binom{1}{0}}) = \gamma_{\binom{1}{0}}. \tag{4.59}$$

Indeed, if $(x,y) \in \gamma_{\binom{1}{0}}$, there exists $t_o \in [0,2\pi)$ such that

$$(x,y) = \theta(t_o,1,0) = (\cos(t_o), \sin(t_o)).$$

Setting $t_m = t_o + 2\pi m$, we see that

$$t_m \to \infty, \quad \text{ as } m \to \infty,$$

and

$$(x,y) = \theta(t_m,1,0), \quad \text{ for all } m \in \mathbf{N},$$

by the fact that sin and cos are periodic functions of period 2π. Consequently,

$$(x,y) = \lim_{m\to\infty} \theta(t_m,1,0),$$

and therefore $(x,y) \in \omega(\gamma_{\binom{1}{0}})$. Thus,

$$\gamma_{\binom{1}{0}} \subseteq \omega(\gamma_{\binom{1}{0}}). \tag{4.60}$$

To see why the reverse inclusion holds, let $(p,q) \in \omega(\gamma_{\binom{1}{0}})$. Then, there exists a sequence of real numbers, (t_m), such that

$$t_m \to \infty, \quad \text{ as } m \to \infty,$$

and

$$\lim_{m\to\infty} \theta(t_m,1,0) = (p,q).$$

Note that $\|\theta(t_m,1,0)\| = 1$ for all m, so that, by continuity of the norm, $\|(p,q)\| = 1$, which shows that $(p,q) \in \gamma_{\binom{1}{0}}$. Consequently,

$$\omega(\gamma_{\binom{1}{0}}) \subseteq \gamma_{\binom{1}{0}}. \tag{4.61}$$

Combining (4.60) and (4.61) fields (4.59).

Example 4.6.8 (Negative Gradient Flows)**.** Let U be an open subset of $\mathbb{R}^N$ which contains the origin, 0. Let $V : U \to \mathbb{R}$ be a C^2 function satisfying $V(x) > 0$ for all $x \in U\backslash\{0\}$ and $V(0) = 0$. Put $F = -\nabla V$ and, for $p \in U$, consider the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = p. \end{cases} \tag{4.62}$$

In this example we show that 0 is an equilibrium point of F and, if F has no equilibrium points in some neighborhood, $\overline{B_r(0)}$, of 0, other than zero, then $\omega(\gamma_p) = \{0\}$ for all $p \in B_r(0)$ which are sufficiently close to 0. We will prove these facts in stages.

(a) The vector 0 is an equilibrium point of the system in (4.62).

Proof: Let $\widehat{w}$ be any unit vector in $\mathbb{R}^N$ and observe that, for any $t \neq 0$,

$$V(t\widehat{w}) - V(0) > 0. \tag{4.63}$$

Dividing the expression in (4.63) by $t > 0$ and letting $t \to 0^+$ yields

$$\nabla V(0) \cdot \widehat{w} \geqslant 0. \tag{4.64}$$

Similarly, dividing the expression in (4.63) by $t < 0$ and letting $t \to 0^-$ yields

$$\nabla V(0) \cdot \widehat{w} \leqslant 0. \tag{4.65}$$

Combining (4.64) and (4.65) we obtain that

$$\nabla V(0) \cdot \widehat{w} = 0, \qquad \text{for all } \widehat{w} \in \mathbb{R}^n \text{ with } \|\widehat{w}\| = 1,$$

which implies that $\nabla V(0) = 0$; hence $F(0) = 0$ and therefore 0 is an equilibrium point of the differential equation in (4.62). ■

(b) Let $p \in U$ and suppose that p is not an equilibrium point of F. Then, the function V is strictly decreasing along γ_p; that is, the function $V(u_p(t))$ decreases with increasing t, where $u_p : J_p \to U$ is the unique solution to the IVP in (4.62) defined on a maximal interval of existence J_p.

Proof: Let $u_p : J_p \to U$ be the solution to IVP (4.62) defined on a maximal interval of existence J_p, where p is not an equilibrium point of F. We then have, by uniqueness, that

$$F(u_p(t)) \neq 0, \qquad \text{for all } t \in J_p \tag{4.66}$$

Observe that, by the Chain Rule,

$$\frac{d}{dt}[V(u_p(t))] \quad = \quad \nabla V(u_p(t)) \cdot u_p'(t),$$

or

$$\frac{d}{dt}[V(u_p(t))] \quad = \quad \nabla V(u_p(t)) \cdot F(u_p(t)), \tag{4.67}$$

since u_p solves the differential equation in (4.62). Consequently, using the assumption that $F = -\nabla V$, we obtain from (4.67) that

$$\frac{d}{dt}[V(u_p(t))] \quad = \quad -\|\nabla V(u_p(t))\|^2, \qquad \text{for all } t \in J_p. \tag{4.68}$$

Thus, if p is not an equilibrium point of F, it follows from (4.68) and (4.66) that V is strictly decreasing along the orbit γ_p as t increases. ■

(c) For every $r > 0$ with $\overline{B_r(0)} \subset U$, such that $\overline{B_r(0)}\backslash\{0\}$ contains no equilibrium points of F, there exists $\delta > 0$ such that, $B_\delta(0) \subset U$ and, for every $p \in B_\delta(0)$, $u_p(t) \in \overline{B_r(0)}$ for all $t \in J_p \cap [0, \infty)$.

Proof: Let $r > 0$ be such that $\overline{B_r(0)} \subset U$ and $\overline{B_r(0)}$ contains no equilibrium points of F other than 0. Put

$$\varepsilon = \inf_{\|x\|=r} V(x). \tag{4.69}$$

By the assumption that $V > 0$ on $U\backslash\{0\}$, the compactness of $\partial B_r(0)$, and the continuity of V we have that $\varepsilon > 0$; thus, since $V(0) = 0$ and V is continuous, there exists $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow V(x) < \varepsilon. \tag{4.70}$$

It follows from the result of part (a) that

$$V(u_p(t)) \leqslant V(p), \qquad \text{for all } t \in J_p \cap [0, \infty). \tag{4.71}$$

Consequently, if $\|p\| < \delta$, it follows from (4.71) and (4.70) that

$$V(u_p(t)) < \varepsilon, \qquad \text{for all } t \in J_p \cap [0, \infty). \tag{4.72}$$

We claim that (4.72) and (4.69) imply that, if $\|p\| < \delta$, then

$$u_p(t) \in \overline{B_r(0)}, \qquad \text{for all } t \in J_p \cap [0, \infty). \tag{4.73}$$

Suppose that (4.73) does not hold. Then there exists $t_1 > 0$ such that $t_1 \in J_p$ and $\|u_p(t_1)\| > r$. By the intermediate value theorem, there exists $\bar{t} \in (0, t_1)$ such that $\|u_p(\bar{t})\| = r$; thus, by (4.69),

$$V(u_p(\bar{t}) \geqslant \varepsilon,$$

which is in direct contradiction with (4.72). Hence, (4.73) must hold true for every $p \in B_\delta(0)$. ■

(d) Let $\delta > 0$ be as obtained in the previous part. Then, for every $p \in B_\delta(0)$, the forward semi–orbit, γ_p^+, is bounded. Deduce therefore that $\omega(\gamma_p) \neq \emptyset$.

Proof: From the result of the previous part, it follows that if $p \in B_\delta(0)$, then

$$u_p(t) \in \overline{B_r(0)}, \quad \text{for all } t \in J_p \cap [0, \infty). \tag{4.74}$$

It then follows from the global existence result proved in the lecture notes that $J_p \cap [0, \infty) = [0, \infty)$; in other words, $u_p(t)$ is defined for all $t \geqslant 0$. Thus, the positive semi–orbit, γ_p^+, is defined an

$$\gamma_p^+ \subseteq \overline{B_r(0)}, \tag{4.75}$$

where we have used (4.74); that is, γ_p^+ is bounded.

It follows from (4.75) that γ_p^+ lies in a compact subset of U; hence, $\omega(\gamma_p) \neq \emptyset$; furthermore,

$$\omega(\gamma_p) \subseteq \overline{B_r(0)}. \tag{4.76}$$

■

(e) Let $r > 0$ be as given in part (c) and $\delta > 0$ as obtained in part (c). Prove that, for any $p \in B_\delta(0)$, $\omega(\gamma_p) = \{0\}$.

Proof: First observe that, if $p = 0$, then $\gamma_p = \{0\}$, since 0 is an equilibrium point of the system. Thus, the statement is true in the case $p = 0$. So, assume for the rest of this argument that $p \in B_\delta(0)\backslash\{0\}$.

Observe that, for $p \in B_\delta(0)\backslash\{0\}$,

$$u_p(t) \neq 0, \quad \text{for all } t \geqslant 0, \tag{4.77}$$

by uniqueness of the solution to the IVP in (4.62).

We next observe that, by (4.68) in part (b) and (4.77), V decreases along γ_p^+ as t increases. We then have that $\lim\limits_{t\to\infty} V(u_p(t))$ exists. We prove that

$$\lim_{t\to\infty} V(u_p(t)) = 0. \tag{4.78}$$

Arguing by contradiction, if (4.78) does not hold true, then there exists $\varepsilon_o > 0$ such that

$$\lim_{t\to\infty} V(u_p(t)) = \varepsilon_o. \tag{4.79}$$

Thus, since $V(u_p(t))$ decreases with t, it follows that

$$V(\theta(t, p)) \geqslant \varepsilon_o, \quad \text{for all } t \geqslant 0, \tag{4.80}$$

where we have written $\theta(t, p)$ for $u_p(t)$. Combining (4.80) with (4.72) in the proof of part (c), we obtain that

$$\varepsilon_o \leqslant V(\theta(t, p)) \leqslant \varepsilon, \quad \text{for all } t \geqslant 0, \tag{4.81}$$

where ε is as given in (4.69).

Define

$$C = \{\theta(t,p) \mid t \geqslant 0 \text{ and } \varepsilon_o \leqslant V(\theta(t,p)) \leqslant \varepsilon\}. \tag{4.82}$$

It follows from the continuity of the functions V and θ and (4.74) that C is a closed subset of a compact set; hence, C is compact. It is also the case that $\theta(t,p) \neq 0$ for all $t \geqslant 0$, since $V(\theta(t,p)) \geqslant \varepsilon_o$ for all $\theta(t,p) \in C$, and $V(x) = 0$ only when $x = 0$. The compactness of C defined in (4.82) and the continuity of ∇V then implies that there exists a positive constant ν such that

$$\frac{d}{dt}[V(\theta(t,p))] \quad \leqslant \quad -\nu, \qquad \text{for all } t \geqslant 0, \tag{4.83}$$

where we have used (4.68). Integrating on both sides of the inequality in (4.83) from 0 to t, we obtain

$$V(\theta(t,p)) \leqslant V(p) - \nu t,$$

which shows that

$$V(\theta(t,p)) < 0, \qquad \text{for } t > \frac{1}{\nu}V(p),$$

which is impossible since $V(x) \geqslant 0$ for all $x \in U$. Hence, (4.79) is not possible, and therefore (4.78) must be true.

Now, given any $\overline{x} \in \omega(\gamma_p)$, there exists a sequence of positive real numbers, (t_m), such that $t_m \to \infty$ as $m \to \infty$ such that

$$\lim_{m\to\infty} \|\theta(t_m,p) - \overline{x}\| = 0. \tag{4.84}$$

It then follows from the continuity of V and (4.84) that

$$\lim_{m\to\infty} V(\theta(t_m,p)) = V(\overline{x}). \tag{4.85}$$

Hence, using (4.78), it follows from (4.85) that

$$V(\overline{x}) = 0, \qquad \text{for all } \overline{x} \in \omega(\gamma_p). \tag{4.86}$$

Therefore, since $V(x) > 0$ for all $x \in U\backslash\{0\}$ and $V(0) = 0$, we obtain from (4.86) that

$$\overline{x} \in \omega(\gamma_p) \Rightarrow \overline{x} = 0;$$

in other words, $\omega(\gamma_p) = \{0\}$. We therefore conclude that $\omega(\gamma_p) = \{0\}$ for all $p \in B_\delta(0)$, which was to be shown. ■

4.7 Cycles and Periodic Solutions

In Example 4.6.7 we saw that the two–dimensional system in (4.58) has an orbit, $\gamma_{(1,0)}$, which is a simple closed curve in $\mathbb{R}^2$,

$$\gamma_{(1,0)} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Definition 4.7.1 (Cycles)**.** An orbit of a dynamical system is a called a cycle when it is a simple closed curve.

Let $\theta \colon \mathbb{R} \times U \to U$ be a dynamical system in an open set $U \subseteq \mathbb{R}^N$. Suppose that for $p \in U$, the orbit of p, γ_p, is a cycle. Then, there exists a positive number T such that $\theta_p(T,p) = p$ and $\theta_p \colon [0,T] \to U$ is a parametrization of γ_p. In other words, $\theta_p \colon [0,T) \to U$ is one–to–one, and $\gamma_p = \theta_p([0,T])$. The function θ_p is said to be periodic with period T.

Definition 4.7.2 (Periodic Solutions)**.** Let $U \subseteq \mathbb{R}^N$ be open and $F \colon U \to \mathbb{R}^N$ be a C^1 vector field. A solution $u \colon \mathbb{R} \to U$ of the differential equation

$$\frac{dx}{dt} = F(x), \tag{4.87}$$

which is not an equilibrium solution, is said to be periodic if there exists a positive number, τ, such that

$$u(t+\tau) = u(t), \qquad \text{for all } t \in \mathbb{R}. \tag{4.88}$$

The smallest positive number, τ, for which (4.88) holds true is called the period of u.

Example 4.7.3. Suppose that the differential equation in (4.87) has a flow, $\theta \colon \mathbb{R} \times U \to U$, and that the orbit, γ_p, for $p \in U$, is a cycle. Let $T > 0$ be such that

$$\theta(T,p) = p, \tag{4.89}$$

$$\theta_p([0,T]) = \gamma_p,$$

and

$$\theta_p \colon [0,T) \to U \text{ is one–to–one}. \tag{4.90}$$

We show that the function

$$\theta_p \colon \mathbb{R} \to U$$

is periodic with period T. To see why this claim is true, observe that, for any $t \in \mathbb{R}$,

$$\theta(t+T,p) = \theta(t,\theta(T,p)) = \theta(t,p),$$

where we have used (4.89). We therefore have that

$$\theta_p(t+T) = \theta_p(t), \qquad \text{for all } t \in \mathbb{R},$$

which shows that θ_p is periodic. To see that T is the period of θ_p, suppose that $0 < \tau < T$, and

$$\theta_p(t+\tau) = \theta_p(t), \quad \text{for all } t \in \mathbb{R}. \tag{4.91}$$

Letting $t = 0$ in (4.91) we obtain that

$$\theta_p(\tau) = \theta_p(0),$$

which contradicts (4.90) since $\tau \in (0, T)$. Hence, cycles in a dynamical system correspond to period solutions.

4.8 Limit Cycles

Example 4.8.1. Consider the two–dimensional system

$$\begin{cases} \dfrac{dx}{dt} & = & -y + \mu x(1 - x^2 - y^2); \\ \\ \dfrac{dy}{dt} & = & x + \mu y(1 - x^2 - y^2), \end{cases} \tag{4.92}$$

where μ is a real parameter.

We would like to understand the dynamics of the system in (4.92). In order to do so, we first look for equilibrium points. We will then look at orbits and compute limit sets of those orbits.

To find the equilibrium points of the system in (4.92), we solve the algebraic system

$$\begin{cases} -y + \mu x(1 - x^2 - y^2) & = & 0; \\ x + \mu y(1 - x^2 - y^2) & = & 0. \end{cases} \tag{4.93}$$

Note that $(0, 0)$ is a solution of the system in (4.93), so that $(0, 0)$ is an equilibrium point. We next see if there are other equilibrium points. So, suppose that (x, y) solves the system in (4.93) and $x^2 + y^2 \neq 0$. We claim that $xy \neq 0$. For suppose that $x = 0$ and $x^2 + y^2 \neq 0$. It then follows from the first equation in (4.93) that $y = 0$, which is impossible. Similarly, if $y = 0$ and $x^2 + y^2 \neq 0$, we obtain from the second equation in (4.93) that $x = 0$, which is impossible. We therefore get that, if (x, y) solves the system in (4.93) and $x^2 + y^2 \neq 0$, then $x \neq 0$ and $y \neq 0$. Thus, let (x, y) be a solution of (4.93) with $x^2 + y^2 \neq 0$. Multiplying the first equation in (4.93) by x and the second equation by y then leads to the equivalent system

$$\begin{cases} -xy + \mu x^2(1 - x^2 - y^2) & = & 0; \\ xy + \mu y^2(1 - x^2 - y^2) & = & 0. \end{cases} \tag{4.94}$$

Adding the equation in (4.94) then leads to

$$\mu(x^2 + y^2)(1 - x^2 - y^2) = 0,$$

which is equivalent to

$$\mu(1 - x^2 - y^2) = 0, \tag{4.95}$$

since we are assuming that $x^2 + y^2 \neq 0$. It follows from (4.95) that either $\mu = 0$ or $x^2 + y^2 = 1$. If $\mu = 0$, then we obtain from (4.93) that $(x, y) = (0, 0)$, which is impossible. Similarly, if $x^2 + y^2 = 1$, we get from (4.95) that $(x, y) = (0, 0)$, a contradiction. We have therefore shown that $(0, 0)$ is the only equilibrium point of the system in (4.92).

We next show that, for $\mu < 0$, if $(p, q) \in B_1(0, 0)$, then $\omega(\gamma_{(p,q)}) = \{(0, 0)\}$. We will proceed as in Example 4.6.8 by considering the function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x, y) = x^2 + y^2, \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

and first showing that if $(p, q) \in B_1(0, 0)$, then $V(u_{(p,q)}(t))$ decreases with increasing t; in other words, V decreases along the orbit, $\gamma_{(p,q)}$. Here, the function $u_{(p,q)} : J_{(p,q)} \to \mathbb{R}^2$ denotes the solution to the system in (4.95) subject to the initial condition:

$$u_p(0) = (p, q), \tag{4.96}$$

where $J_{(p,q)}$ is the maximal interval of existence. In order to show that $V(u_{(p,q)}(t))$ decreases with increasing t, first compute

$$\begin{aligned}
\frac{d}{dt}\left[V(u_{(p,q)}(t))\right] &= \nabla V(u_{(p,q)}(t)) \cdot u'_{(p,q)}(t) \\
&= 2x(-y + \mu x(1 - x^2 - y^2)) + 2y(x + \mu y(1 - x^2 - y^2)) \\
&= 2\mu(x^2 + y^2)(1 - (x^2 + y^2)).
\end{aligned}$$

We therefore have that

$$\frac{d}{dt}\left[V(u_{(p,q)}(t))\right] = 2\mu V(u_{(p,q)}(t))[1 - V(u_{(p,q)}(t))]. \tag{4.97}$$

Setting $v(t) = V(u_{(p,q)}(t))$, for $t \in J_{(p,q)}$, we can re-write the differential equation in (4.97) as

$$\frac{dv}{dt} = 2\mu v(1 - v). \tag{4.98}$$

Note from (4.98) that $\dfrac{dv}{dt} < 0$, whenever $0 < v < 1$, for the case $\mu < 0$. Thus, if $(p, q) \in B_1(0, 0)$, it follows from (4.96) that

$$\|u_{(p,q)}(t)\|^2 \leqslant \|(p, q)\|^2 < 1, \quad \text{for all } t \in J_{(p,q)} \cap [0, \infty),$$

from which we get that

$$u_{(p,q)}(t) \in B_1(0, 0), \quad \text{for all } t \in J_{(p,q)} \cap [0, \infty), \text{ and } (p, q) \in B_1(0, 0). \tag{4.99}$$

It follows from the Proposition 2.3.9, on page 23 in these notes, that $u_{(p,q)}(t)$ is defined for all $t \geqslant 0$. Furthermore, we obtain from (4.99) that

$$\gamma^+_{(p,q)} \subset B_1(0, 0).$$

It then follows from Proposition 4.6.6 that, if $(p,q)\in B_1(0,0)$, then $\omega(\gamma_{(p,q)})\neq\emptyset$ and

$$\omega(\gamma_{(p,q)})\subseteq\overline{B_1(0,0)}.$$

We next show that

$$\omega(\gamma_{(p,q)})=\{(0,0)\},\ \text{for all } (p,q)\in B_1(0,0). \tag{4.100}$$

In order to prove (4.100), we solve (4.98) by separation of variables and partial fractions to obtain

$$v(t)=\frac{v_oe^{2\mu t}}{1-v_o+v_oe^{2\mu t}},\qquad \text{for all } t\in\mathbb{R}, \tag{4.101}$$

where $v_o=p^2+q^2<1$. We then see from (4.101) that, if $(p,q)\in B_1(0,0)$ and $\mu<0$, then

$$\lim_{t\to\infty}v(t)=0,$$

or

$$\lim_{t\to\infty}V(u_{(p,q)}(t))=0,$$

from which we get that

$$\lim_{t\to\infty}\|\theta(t,p,q)\|=0,\qquad \text{whenever } (p,q)\in B_1(0,0), \tag{4.102}$$

where we have written $\theta(t,p,q)$ for $u_{(p,q)}(t)$, for all $t\geqslant 0$. Finally, to show (4.100) holds true, assume that $(p,q)\in B_1(0,0)$ and let $(\overline{x},\overline{y})\in\omega(\gamma_{(p,q)})$. Then, there exists a sequence of positive numbers, $(t_m0$, such that $t_m\to\infty$ and

$$\lim_{m\to\infty}\theta(t_m,p,q)=(\overline{x},\overline{y}). \tag{4.103}$$

It then follows from (4.102), (4.103), and the continuity of the flows and of the norm that

$$\|(\overline{x},\overline{y})\|=0,$$

from which get that $(\overline{x},\overline{y})=(0,0)$ and (4.100) follows.

Example 4.8.2 (Continuation of Example 4.8.1)**.** In this example we show that if $\mu<0$ in system (4.92), and $(p,q)\in B_1(0,0)\backslash\{(0,0)\}$, then $\alpha(\gamma_{(p,q)})=S^1$, where $S^1=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2=1\}$.

It follows from (4.101) that, if $0<v_o=p^1+q^2<1$, then

$$V(u_{(p,q)}(t))<1,\qquad \text{for all } t\in J_{(p,q)}.$$

Thus, $u_{(p,q)}(t)\in B_1(0,0)$ for all $t\in J_{(p,q)}$. The global existence results proved in in these notes then imply that $u_{(p,q)}(t)$ is defined for all $t\in\mathbb{R}$. Furthermore,

$$\gamma^-_{(p,q)}\subset\overline{B_1(0,0)};$$

it then follows from Proposition 4.6.6 that $\alpha(\gamma_{(p,q)}) \neq \emptyset$ and

$$\alpha(\gamma_{(p,q)}) \subset \overline{B_1(0,0)}.$$

Next, we show that

$$\alpha(\gamma_{(p,q)}) \subseteq \partial B_1(0,0) = S^1, \quad \text{for } 0 < p^2 + q^2 < 1. \tag{4.104}$$

Let $(\overline{x}, \overline{y}) \in \alpha(\gamma_{(p,q)})$; then, there exists a sequence of negative numbers, (t_m), such that

$$t_m \to -\infty \text{ as } m \to \infty, \tag{4.105}$$

and

$$\lim_{m\to\infty} \theta(t_m, p, q) = (\overline{x}, \overline{y}). \tag{4.106}$$

It follows from (4.101) and (4.105) that, if $\mu < 0$ and $0 < v_o < 1$, then

$$\lim_{m\to\infty} v(t_m) = \lim_{m\to\infty} \frac{v_o}{(1 - v_o)e^{-2\mu t_m} + v_o} = 1,$$

from which we get that

$$\lim_{m\to\infty} \|\theta(t_m, p, q)\| = 1. \tag{4.107}$$

It follows from (4.106), (4.107), and the continuity of the flow and the norm that

$$\|(\overline{x}, \overline{y})\| = 1,$$

which proves (4.104). In order to prove the reverse inclusion,

$$S^1 \subseteq \alpha(\gamma_{(p,q)}), \quad \text{for } 0 < p^2 + q^2 < 1, \tag{4.108}$$

let $(\overline{x}, \overline{y}) \in S^1$ then $\overline{x}^2 + \overline{y}^2 = 1$, and so there exists a unique $\overline{\varphi} \in [0, 2\pi)$ such that

$$\overline{x} = \cos\overline{\varphi} \quad \text{and} \quad \overline{y} = \sin\overline{\varphi}. \tag{4.109}$$

For $(p, q) \in B_1(0,0)$ with $p^2 + q^2 \neq 0$, denote $\theta(t, p, q)$ by $(x(t), y(t))$ and put

$$r(t) = \sqrt{[x(t)]^2 + [y(t)]^2}, \quad \text{for } t \in \mathbb{R},$$

so that

$$r(t) = \sqrt{V(\theta(t, p, q))} = \sqrt{v(t)}, \quad \text{for all } t \in \mathbb{R}.$$

It then follows from

$$r(t) = \frac{v_o^{1/2}}{\sqrt{(1 - v_o)e^{-2\mu t} + v_o}}, \quad \text{for all } t \in \mathbb{R}. \tag{4.110}$$

Next, put

$$\varphi(t) = \arctan\left(\frac{y(t)}{x(t)}\right), \quad \text{for } t \in \mathbb{R},$$

so that

$$\tan(\varphi(t)) = \frac{y(t)}{x(t)}, \quad \text{for } t \in \mathbb{R}. \tag{4.111}$$

Differentiation on both sides of the equation in (4.111) with respect to t then yields

$$\sec^2\varphi \frac{d\varphi}{dt} = -\frac{y}{x^2}\frac{dx}{dt} + \frac{1}{x}\frac{dy}{dt},$$

or

$$\left(1 + \frac{y^2}{x^2}\right)\frac{d\varphi}{dt} = -\frac{y}{x^2}\frac{dx}{dt} + \frac{1}{x}\frac{dy}{dt},$$

from which we get that

$$\frac{d\varphi}{dt} = -\frac{y}{x^2+y^2}\frac{dx}{dt} + \frac{x}{x^2+y^2}\frac{dy}{dt}. \tag{4.112}$$

Since $(x(t), y(t))$ solves the system in (4.8.1), we can substitute the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in (4.8.1) into (4.112) to obtain

$$\frac{d\varphi}{dt} = -\frac{-y^2 + \mu xy(1-x^2-y^2)}{x^2+y^2} + \frac{x^2 + \mu xy(1-x^2-y^2)}{x^2+y^2},$$

which leads to

$$\frac{d\varphi}{dt} = 1. \tag{4.113}$$

We can solve (4.113) to obtain

$$\varphi(t) = t + \varphi_o, \text{ for all } t \in \mathbb{R},$$

and some $\varphi_o \in [0, 2\pi)$. We then have that

$$\theta(t,p,q) = r(t)(\cos(t+\varphi_o), \sin(t+\varphi_o)), \quad \text{for all } t \in \mathbb{R}, \tag{4.114}$$

where $r(t)$ is given in (4.110). Define the sequence of real numbers

$$t_m = \overline{\varphi} - \varphi_o - 2m\pi, \quad \text{for all } m \in \mathbf{N}. \tag{4.115}$$

Then

$$t_m \to -\infty \quad \text{as} \quad m \to \infty. \tag{4.116}$$

Substituting t in (4.114) for t_m given in (4.115) we obtain from (4.114) that

$$\theta(t_m,p,q) = r(t_m)(\cos\overline{\varphi}, \sin\overline{\varphi}), \quad \text{for all } m \in \mathbf{N}, \tag{4.117}$$

where we have used the 2π–periodicity of sin and cos. Thus, using the definition of $r(t)$ in (4.110) and (4.116), we obtain from (4.117) that

$$\lim_{m\to\infty} \theta(t_m,p,q) = (\cos\overline{\varphi}, \sin\overline{\varphi}) = (\overline{x}, \overline{y}),$$

for $0 < p^2+q^2 < 1$, where we have used (4.109); thus, $(\overline{x}, \overline{y}) \in \omega(\gamma_{(p,q)})$, provided that $0 < p^2 + q^2 < 1$. We have therefore established the inclusion in (4.108). Combining (4.108) with (4.104) yields that, if $\mu < 0$ and $\gamma_{(p,q)}$ is an orbit of the system in (4.92) with $0 < p^2 + q^2 < 1$, then

$$\alpha(\gamma_{(p,q)}) = S^1.$$

Example 4.8.3 (Continuation of Example 4.8.2)**.** Assume $(p, q) \in S^1 = \partial B_1(0, 0)$ and that $u_{(p,q)} \colon J_{(p,q)} \to \mathbb{R}^2$ is the unique solution to the system in (4.92) subject to the initial condition:

$$x(0) = p \quad \text{ and } \quad y(0) = q, \tag{4.118}$$

where

$$p^2 + q^2 = 1.$$

Then, there exists $\varphi_o \in [0, 2\pi)$ such that

$$p = \cos\varphi_o \quad \text{ and } \quad q = \sin\varphi_o. \tag{4.119}$$

We show that

$$u_{(p,q)}(t) = (\cos(t + \varphi_o), \sin(t + \varphi_o)), \quad \text{ for all } t \in \mathbb{R}. \tag{4.120}$$

Put

$$v(t) = (\cos(t + \varphi_o), \sin(t + \varphi_o)), \quad \text{ for all } t \in \mathbb{R}. \tag{4.121}$$

First note that, in view of (4.119), the function $v = v(t)$ given in (4.122) satisfies the initial condition in (4.118). Writing $x(t) = \cos(t+\varphi_o)$ and $y(t) = \sin(t+\varphi_o)$ for all $t \in \mathbb{R}$, we compute that

$$x'(t) = -\sin(t + \varphi_o) = -y(t), \quad \text{ for all } t \in \mathbb{R},$$

and

$$y'(t) = \cos(t + \varphi_o) = x(t), \quad \text{ for all } t \in \mathbb{R},$$

so that, since

$$[x(t)]^2 + [y(t)]^2 = 1, \quad \text{ for all } t \in \mathbb{R},$$

we see that $x = x(t)$ and $y = y(t)$ solve the system in (4.92). By uniqueness of the solution to the IVP in (4.92) and (4.118), we have that

$$v(t) = u_{(p,q)}(t), \quad \text{ for all } t \in \mathbb{R},$$

so that the assertion in (4.120) follows.

We then have that if $(p, q) \in S^1$, $\alpha(\gamma_{(p,q)}) = \omega(\gamma_{(p,q)}) = S^1$ (see Example 4.6.7 on page 61 in these notes).

Example 4.8.4 (Continuation of Example 4.8.3)**.** Assume that $p^2 + q^2 > 1$. Let $u_{(p,q)} \colon J_{(p,q)} \to \mathbb{R}^2$ be the solution to the IVP in (4.92), for $\mu < 0$, subject to the initial condition

$$u(0) = (p, q). \tag{4.122}$$

As in Example 4.8.1, let $V(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$, and put

$$v(t) = V(u_{(p,q)}(t)), \quad \text{for all } t \in J_{(p,q)}.$$

Then, v satisfies the first order differential equation in (4.98), which can be solved to yield

$$v(t) = \frac{v_o}{v_o - (v_o - 1)e^{-2\mu t}}, \quad \text{for } t < t_1, \tag{4.123}$$

for the case $v > 1$, where

$$v_o = p^2 + q^2,$$

and

$$t_1 = \frac{1}{2\mu} \ln\left(\frac{v_o - 1}{v_o}\right) > 0, \tag{4.124}$$

since $\mu < 0$. Observe that for $t < t_1$,

$$-2\mu t < -2\mu t_1, \tag{4.125}$$

since $\mu < 0$. Exponentiating on both sides of (4.125) yields

$$e^{-2\mu t} < \frac{v_o}{v_o - 1}, \tag{4.126}$$

where we have used the definition of t_1 in (4.124). Consequently, since $v_o > 1$, we obtain from (4.126) that

$$v_o - (v_o - 1)e^{-2\mu t} > 0 \quad \text{for all } t < t_1,$$

where t_1 is given in (4.124); hence, $v(t)$ in (4.124) is indeed defined for all $t < t_1$. Thus, $J_{(p,q)} = (-\infty, t_1)$, and so the ω–limit set of $\gamma_{(p,q)}$ is not defined for the case $\mu < 0$ and $p^2 + q^2 > 1$. On the other hand, since by virtue of (4.123),

$$\lim_{t\to-\infty} v(t) = \lim_{t\to-\infty} \frac{v_o}{v_o - (v_o - 1)e^{-2\mu t}} = 1, \tag{4.127}$$

for $\mu < 0$, we can prove that

$$\alpha(\gamma_{(p,q)}) = S^1, \quad \text{for } p^2 + q^2 > 1. \tag{4.128}$$

In order to prove (4.128), first note that, by virtue of (4.98), if $\mu < 0$ and $v_o = p^2 + q^2 > 1$, then $v(t)$ increases with increasing t; so that,

$$v(t) < v_o, \quad \text{for all } t < 0,$$

from which we get that

$$\|u_{(p,q)}(t)\| < \sqrt{v_o}, \quad \text{for all } t < 0.$$

Thus, $\gamma^-_{(p,q)}$ is bounded and therefore, by Proposition 4.6.6, $\alpha(\gamma_{(p,q)}) \neq \emptyset$ for $p^2+q^2>1$, provided that $\mu<0$.

Next, let $(\overline{x},\overline{y}) \in \alpha(\gamma_{(p,q)})$. Then there exists a sequence, (t_m), of negative numbers such that

$$t_m \to -\infty \quad \text{as} \quad m \to \infty, \tag{4.129}$$

and

$$\lim_{m\to\infty} u_{(p,q)}(t_m) = (\overline{x},\overline{y}). \tag{4.130}$$

Thus, by the continuity of the function V we obtain from (4.130) that

$$\lim_{m\to\infty} V(u_{(p,q)}(t_m)) = V(\overline{x},\overline{y}),$$

or

$$\lim_{m\to\infty} v(t_m) = V(\overline{x},\overline{y}). \tag{4.131}$$

Thus, using (4.127) and (4.129), we obtain from (4.131) that

$$V(\overline{x},\overline{y}) = 1,$$

which implies that $(\overline{x},\overline{y}) \in S^1$. We have therefore shown that

$$\alpha(\gamma_{(p,q)}) \subseteq S^1, \quad \text{for } p^2+q^2>1, \text{ and } \mu<0. \tag{4.132}$$

Next, let $(\overline{x},\overline{y}) \in S^1$ and $\overline{\varphi} \in [0,2\pi)$ be such that

$$\overline{x} = \cos\overline{\varphi} \quad \text{and} \quad \overline{y} = \sin\overline{\varphi}. \tag{4.133}$$

Similarly, let $\varphi_o \in [0,2\pi)$ be such that

$$\frac{p}{\sqrt{v_o}} = \cos\varphi_o \quad \text{and} \quad \frac{q}{\sqrt{v_o}} = \sin\varphi_o. \tag{4.134}$$

We then have that

$$u_{(p,q)}(t) = r(t)(\cos(t+\varphi_o), \sin(t+\varphi_o)), \text{ for all } t<t_1, \tag{4.135}$$

where

$$r(t) = \sqrt{v(t)}, \quad \text{for all } t<t_1, \tag{4.136}$$

where $v(t)$ is given in (4.123).

Put

$$t_m = -2m\pi + \overline{\varphi} - \varphi_o, \quad \text{for } m=1,2,3,\ldots; \tag{4.137}$$

then,

$$t_m \to -\infty \quad \text{as } m\to\infty. \tag{4.138}$$

Substituting t_m in (4.137) for t in the expression defining $u_{(p,q)}(t)$ in (4.135) yields

$$u_{(p,q)}(t_m) = r(t_m)(\cos(\overline{\varphi}), \sin(\overline{\varphi})), \text{ for } m=1,2,3,\ldots, \tag{4.139}$$

where we have used the 2π–periodicity of sin and cos. Thus, using (4.138), (4.127) and (4.133), we obtain from (4.139) that

$$\lim_{m\to\infty} u_{(p,q)}(t_m) = (\overline{x}, \overline{y}),$$

which shows that $(\overline{x}, \overline{y}) \in \alpha(\gamma_{(p,q)})$; so that

$$S^1 \subseteq \alpha(\gamma_{(p,q)}), \qquad \text{for } p^2 + q^2 > 1, \text{ and } \mu < 0. \tag{4.140}$$

Combining (4.132) with (4.140) yields (4.128).

In Examples 4.8.2 through 4.8.4 in this section, we have shown that S^1 is a cycle of the system in (4.92) with the property that if either $0 < p^2 + q^2 < 1$ or $p^2 + q^2 > 1$, then $\alpha(\gamma_{(p,q)}) = S^1$, for $\mu < 0$. We then say that, for $\mu < 0$, the orbit S^1 is a limit cycle of the system in (4.92).

Definition 4.8.5 (Limit Cycles)**.** Let U be an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. A closed orbit, γ, of the system

$$\frac{dx}{dt} = F(x)$$

is said to be a limit cycle of the system if there exists a neighborhood, $V_\gamma \subset U$, of the orbit γ such that for all $p \in V_\gamma$ either $\omega(\gamma_p) = \gamma$ or $\alpha(\gamma_p) = \gamma$.

In Examples 4.8.2 and 4.8.4 we saw that if $V = \mathbb{R}^N\backslash\{0\}$, then $\alpha(\gamma_{(p,q)}) = S^1$ if $(p, q) \in V\backslash S^1$. In Chapter 5 of these notes we will study criteria that can be used to determine whether a given, general, two–dimensional system has limit cycle.

4.9 Liapunov Stability

Let U denote an open subset of $\mathbb{R}^N$ and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. Suppose that $\overline{x}$ is an equilibrium point of the system

$$\frac{dx}{dt} = F(x). \tag{4.141}$$

Assume also that there exists $\overline{r} > 0$ such that $\overline{B_{\overline{r}}(\overline{x})} \subset U$, and $\overline{B_{\overline{r}}(\overline{x})}\backslash\{\overline{x}\}$ contains no equilibrium points of F; in other words, $\overline{x}$ is an isolated equilibrium point of F in U. In this section we are interested in conditions that will guarantee that if a solution of (4.141) begins near the equilibrium point, $\overline{x}$, then it will remain near $\overline{x}$ for all $t > 0$. This is the concept of stability which we make precise in the following definition.

Definition 4.9.1 (Liapunov Stability)**.** Let $\overline{x}$ be an isolated equilibrium point of the system in (4.141) and let $\overline{r} > 0$ be such that $\overline{B_{\overline{r}}(\overline{x})} \subset U$, and $\overline{B_{\overline{r}}(\overline{x})}\backslash\{\overline{x}\}$ contains no equilibrium points of F. We say that $\overline{x}$ is stable if, for every $r \in (0, \overline{r})$, there exists $\delta > 0$ such that, if $\|p - \overline{x}\| < \delta$, then the solution, $u_p \colon J_p \to U$, to the IVP

$$\begin{cases} \dfrac{dx}{dt} &= F(x); \\ x(0) &= p, \end{cases} \tag{4.142}$$

exists for all $t > 0$, and there exists $t_1 > 0$ such that

$$u_p(t) \in B_r(\overline{x}), \quad \text{for } t \geqslant t_1.$$

Example 4.9.2. In part (c) of Example 4.6.8 on page 63, we saw that for $F = -\nabla V$, where $V \colon U \to \mathbb{R}$ is a C^2 function satisfying $V(x) > 0$ for all $x \in U\backslash\{0\}$ and $V(0) = 0$, and if 0 is an isolated equilibrium point, then, for every $r > 0$ with $\overline{B_r(0)} \subset U$, such that $\overline{B_r(0)}\backslash\{0\}$ contains no equilibrium points of F, there exists $\delta > 0$ such that, $B_\delta(0) \subset U$ and, for every $p \in B_\delta(0)$, $u_p(t) \in \overline{B_r(0)}$ for all $t \in J_p \cap [0, \infty)$, where $u_p \colon J_p \to U$ is the unique solution to the IVP in (4.142). This implies that $u_p(t)$ is defined for all $t > 0$ and

$$u_p(t) \in \overline{B_r(0)}, \quad \text{for all } t \geqslant 0.$$

Hence, $\overline{x} = 0$ is stable according to Definition 4.9.1.

Definition 4.9.3 (Asymptotic Stability)**.** An isolated equilibrium point, $\overline{x}$, of the system in (4.141) is said to be asymptotically stable if $\overline{x}$ is stable and there exists $\delta > 0$ such that, if $\|p - \overline{x}\| < \delta$, then $\omega(\gamma_p) = \{\overline{x}\}$.

Example 4.9.4. Let $F = -\nabla V$, where V is as given in Example 4.6.8 on page 63. In parts (d) and (e) of that example we showed that there exists $\delta > 0$ such that, for any $p \in B_\delta(0)$, $\omega(\gamma_p) = \{0\}$. In other words, $\overline{x} = 0$ is asymptotically stable.

Definition 4.9.5 (Unstable Equilibrium Points)**.** An isolated equilibrium point, $\overline{x}$, of the system in (4.141) which is not stable is said to be unstable.

Example 4.9.6. Consider the system

$$\begin{cases} \dfrac{dx}{dt} &= y + \mu x^3; \\ \dfrac{dy}{dt} &= -x + \mu y^3, \end{cases} \tag{4.143}$$

where $\mu > 0$. We show that $(0, 0)$ is an unstable equilibrium point of the system in (4.143).

Solution: First note that $(0,0)$ is the only equilibrium point of the system in (4.143). Indeed, suppose that $(\overline{x},\overline{y})$ is an equilibrium point of the system in (4.143) with $(\overline{x},\overline{y}) \neq (0,0)$. We then have that

$$\begin{cases} \overline{y} + \mu\overline{x}^3 = 0; \\ -\overline{x} + \mu\overline{y}^3 = 0. \end{cases} \tag{4.144}$$

We see that $\overline{x} \neq 0$ and $\overline{y} \neq 0$. For, suppose that $\overline{x} = 0$, the from the second equation in (4.144), we get that $\overline{y} = 0$ since $\mu > 0$, which is impossible since we are assuming that $(\overline{x},\overline{y}) \neq (0,0)$. Similarly, $\overline{y}$ cannot be 0. Next, multiply the first equation in (4.144) by $\overline{x}$ and the second equation by $\overline{y}$ to get that

$$\begin{cases} \overline{xy} + \mu\overline{x}^4 = 0; \\ -\overline{xy} + \mu\overline{y}^4 = 0. \end{cases} \tag{4.145}$$

Adding the two equations in (4.145) we get that

$$\mu(\overline{x}^4 + \overline{y}^4) = 0,$$

which yields that

$$\overline{x}^4 + \overline{y}^4 = 0, \tag{4.146}$$

since $\mu > 0$. Note that (4.146) is impossible for $(\overline{x},\overline{y}) \neq (0,0)$. We have therefore shown that $(0,0)$ is an isolated equilibrium point of the system in (4.143).

Next, we show that $\mu > 0$ implies that $(0,0)$ is unstable.

We will prove that, for every $r > 0$ and every $\delta \in (0,r)$, there exists $(p,q) \in B_\delta(0,0)$, such that $(p,q) \neq (0,0)$ and $\|u_{(p,q)}(t_1)\| > r$ for some $t_1 > 0$. In fact, let $(p,q) \neq (0,0)$ be such that $p^2 + q^2 < \delta$ and let $u_{(p,q)} \colon J_{(p,q)} \to \mathbb{R}^2$ denote the solution to the system in (4.143) subject to the initial condition

$$(x(0), y(0)) = (p,q). \tag{4.147}$$

Suppose by way of contradiction that

$$u_{(p,q)}(t) \in \overline{B_r(0,0)}, \quad \text{for all } t \in J_{(p,q)} \cap [0,+\infty). \tag{4.148}$$

It follows from (4.148) and Proposition 2.3.9 on page 23 in this notes that $u_{(p,q)}$ is defined for all $t \geqslant 0$ and

$$u_{(p,q)}(t) \in \overline{B_r(0,0)}, \quad \text{for all } t \geqslant 0. \tag{4.149}$$

Define $V(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$ and put

$$u_{(p,q)}(t) = (x(t), y(t)), \quad \text{for all } t \geqslant 0.$$

Applying the Chain Rule we obtain that

$$\frac{d}{dt}\left[V(u_{(p,q)}(t))\right] = 2\mu([x(t)]^4 + [y(t)]^4), \quad \text{for all } t \geqslant 0. \tag{4.150}$$

Note also that, since $(p,q) \neq (0,0)$, it follows from uniqueness that

$$u_{(p,q)}(t) \neq (0,0), \quad \text{for all } t \geqslant 0. \tag{4.151}$$

In view of (4.151) and (4.150), we obtain from (4.150) that

$$\frac{d}{dt}\left[V(u_{(p,q)}(t))\right] > 0, \quad \text{for all } t \geqslant 0,$$

since $\mu > 0$. Consequently, $V(u_{(p,q)}(t))$ is increasing in t. Thus,

$$V(u_{(p,q)}(t)) \geqslant V(p,q), \quad \text{for all } t \geqslant 0. \tag{4.152}$$

Using the definition of V and (4.149), we obtain from (4.152) that

$$r_o \leqslant \|u_{(p,q)}(t)\| \leqslant r, \quad \text{for all } t \geqslant 0, \tag{4.153}$$

where $r_o = \|(p,q)\|$. Since we are assuming that $(p,q) \neq (0,0)$, we have that $r_o > 0$. Put

$$\nu = \min_{r_o \leqslant \|(x,y)\| \leqslant r} 2\mu(x^4 + y^4). \tag{4.154}$$

Then, $\nu > 0$ since $r_o > 0$ and $x^4 + y^4 = 0$ if and only if $(x,y) = (0,0)$. It follows from (4.153), (4.150) and (4.154) that

$$\frac{d}{dt}\left[V(u_{(p,q)}(t))\right] \geqslant \nu, \quad \text{for all } t \geqslant 0. \tag{4.155}$$

Integrating the inequality in (4.155) from 0 to t then yields

$$V(u_{(p,q)}(t)) \geqslant V(p,q) + vt, \quad \text{for all } t \geqslant 0,$$

which implies that

$$\|u_{(p,q)}(t)\| \to \infty \quad \text{as } t \to \infty,$$

which contradicts (4.149). This contradiction shows that $(0,0)$ is an unstable equilibrium point for the system in (4.143) for $\mu > 0$. □

The arguments used to prove stability in Example 4.6.8 on page 63 and in Example 4.9.6 to prove instability are instances of a more general procedure developed by Liapunov. This procedure is described in Chapter X of [Hal09] and presented as Liapunov's direct methods. We will describe the Liapunov technique in the remainder of this section and apply to some two–dimensional systems in the next chapter.

The main object in Liapunov's approach to stability is a C^1 function, $V : U \to \mathbb{R}$, which is positive definite in a neighborhood, $\Omega \subset \overline{\Omega} \subset U$ of the origin which decreases along orbits as t increases. We will make these ideas precise in what follows.

Definition 4.9.7 (Lie Derivative)**.** Let $F\colon U \to \mathbb{R}^N$ be a C^1 vector field on a open subset, U, of $\mathbb{R}^N$. Given a C^1 function, $V\colon U \to \mathbb{R}$, we define the derivative of V along orbits of the system

$$\frac{dx}{dt} = F(x) \tag{4.156}$$

to be the map $\dot{V}\colon U \to \mathbb{R}$ given by

$$\dot{V}(x) = \nabla V \cdot F(x), \qquad \text{for all } x \in U. \tag{4.157}$$

Remark 4.9.8. Observe that if $u\colon J \to U$ is a solution to the differential equation in (4.156) defined on a maximal interval of existence, J, then $\dot{V}(u(t))$ gives the rate of change of the function V along the orbit determined by u.

Definition 4.9.9 (Liapunov Function)**.** Let U be an open subset of $\mathbb{R}^N$ and Ω be a open subset of U with $\overline{\Omega} \subset U$. A C^1 function, $V\colon U \to \mathbb{R}$, is said to be a Liapunov function of the system in (4.156) in on the set Ω if and only if

$$\dot{V}(x) \leqslant 0, \qquad \text{for all } x \in \Omega. \tag{4.158}$$

Example 4.9.10. If $V\colon U \to \mathbb{R}$ is a C^2 function, and $F = -\nabla V$, then V is a Liapunov function for the system in (4.156) since, in this case,

$$\dot{V}(x) = \nabla V(x) \cdot F(x) = -\|\nabla f(x)\|^2 \leqslant 0, \qquad \text{for all } x \in U.$$

Definition 4.9.11 (Positive Definite Functions)**.** Let $\Omega \subset \overline{\Omega} \subset U$ be a neighborhood of 0. Let $V\colon U \to \mathbb{R}$ be a continuous function satisfying $V(0) = 0$. We say that V is positive semi–definite in $\overline{\Omega}$ if $V(x) \geqslant 0$ for all $x \in \overline{\Omega}$. We say that V is positive definite in $\overline{\Omega}$ if $V(x) > 0$ for all $x \in \overline{\Omega}\backslash\{0\}$ and $V(0) = 0$.

Definition 4.9.12 (Negative Definite Functions)**.** Let $\Omega \subset \overline{\Omega} \subset U$ be a neighborhood of 0. Let $V\colon U \to \mathbb{R}$ be a continuous function satisfying $V(0) = 0$. We say that V is negative semi–definite in Ω if $-V$ is positive semi–definite We say that V is negative definite in Ω if $-V$ positive definite.

We are now ready to state the first Liapunov stability theorem of this section.

Theorem 4.9.13 (Liapunov Stability Theorem)**.** *Let $\overline{x} = \{0\}$ be an isolated equilibrium point of the system in (4.156). Suppose that the system in (4.156) has a Liapunov function, V, in a neighborhood $\Omega \subset \overline{\Omega} \subset U$ of 0. Assume also that V is positive definite in Ω. Then $\overline{x} = 0$ is a stable equilibrium point of the system in (4.156). In addition, if $\dot{V}$ is negative definite in Ω, then $\omega(\gamma_p) = \{0\}$ for all $p \in \Omega$.*

The argument to prove Theorem 4.9.13 has been outlined in Example 4.6.8 for the case of negative gradient flows.

We also state here an instability criterion.

Theorem 4.9.14 (Instability Criterion)**.** Let $\overline{x} = \{0\}$ be an isolated equilibrium point of the system in (4.156). Let Ω be a neighborhood of 0, and suppose that $V : U \to \mathbb{R}$ is a C^1 function such that $\dot{V}$ is positive definite in $\overline{\Omega}$. Suppose that for every $\delta > 0$, there exists $p \in B_\delta(0)\backslash\{0\}$ such that $V(p) > 0$. Then, $\overline{x} = 0$ is unstable.

The proof of Theorem 4.9.14 is outlined in Example 4.143.

We end this section with the following extension of the Liapunov stability theorem, which is presented as Theorem 1.3 in [Hal09, pg. 316].

Theorem 4.9.15 (The "S&M Theorem")**.** Suppose that the system in (4.156) has a Liapunov function, V, in an open subset, Ω, of U with $\overline{\Omega} \subset U$. Let

$$S = \{x \in \overline{\Omega} \mid \dot{V}(x) = 0\}, \tag{4.159}$$

and denote by M the largest invariant set of the system in (4.156) which is contained in S. Let $p \in \Omega$ and suppose that γ_p^+ is bounded and contained in Ω. Then, $\omega(\gamma_p) \subseteq M$.

Proof: Since γ_p^+ is bounded, it follows from Proposition 4.6.6 on page 58 in these notes that $\omega(\gamma_p)$ is nonempty, compact and connected. It is also the case that the solution, $u_p : J_p \to U$, to the equation in (4.156) subject to the initial condition

$$x(0) = p \tag{4.160}$$

exists for all $t \geqslant 0$ and

$$u_p(t) \in \Omega, \qquad \text{for all } t \geqslant 0. \tag{4.161}$$

Next, use the assumption that V is a Liapunov function in $\overline{\Omega}$ to see that

$$\frac{d}{dt}[V(u_p(t))] = \dot{V}(u_p(t)) \leqslant 0, \qquad \text{for all } x \in \overline{\Omega}, \tag{4.162}$$

where we have used (4.158) and (4.161). It follows from (4.162) that $V(u_p(t))$ is monotone non–increasing with t.

We now see that $V(u_p(t))$ is also bounded below for $t \geqslant 0$. Suppose, by way of contradiction, that there exists a sequence of positive numbers, (t_m), such that

$$t_m \to \infty \quad \text{as} \quad m \to \infty$$

and

$$\lim_{m\to\infty} V(u_p(t_m)) = -\infty. \tag{4.163}$$

The hypotheses that γ_p^+ is bounded and $\gamma_p^+ \subset \Omega$ imply that there exists a subsequence, (t_{m_k}), of (t_m) such that

$$\lim_{k\to\infty} u_p(t_{m_k}) = \overline{y}, \tag{4.164}$$

for some $\overline{y} \in \overline{\Omega}$. It follows from (4.164) and the continuity of V that

$$\lim_{k\to\infty} V(u_p(t_{m_k})) = V(\overline{y}), \tag{4.165}$$

where $V(\overline{y}) \in \mathbb{R}$, since $\overline{y} \in \overline{\Omega} \subset U$. On the other hand, by virtue of (4.163),

$$\lim_{k\to\infty} V(u_p(t_{m_k})) = -\infty,$$

which contradicts (4.165). Hence, $V(u_p(t))$ must be bounded below.

We therefore have that $V(u_p(t))$ is monotone non–increasing as t increases, and bounded from below for $t \geqslant 0$. Consequently, $\lim\limits_{t\to\infty} V(u_p(t))$ exists. Set

$$\lim_{t\to\infty} V(u_p(t)) = c. \tag{4.166}$$

We show that

$$V(\overline{y}) = c, \quad \text{for all } \overline{y} \in \omega(\gamma_p). \tag{4.167}$$

Let $\overline{y} \in \omega(\gamma_p)$; then there exists a sequence of positive numbers, (t_m), such that

$$t_m \to \infty \quad \text{as} \quad m \to \infty \tag{4.168}$$

and

$$\lim_{m\to\infty} u_p(t_m) = \overline{y}. \tag{4.169}$$

Then, using the continuity of V, we obtain from (4.169) that

$$\lim_{m\to\infty} V(u_p(t_m)) = V(\overline{y}). \tag{4.170}$$

Hence, in view of (4.168), we see that (4.167) follows from (4.166) and (4.170). Since, $\omega(\gamma_p)$ is invariant under the flow of F by virtue of Proposition 4.6.3 on page 4.6.3 of these notes, we see from (4.167) that

$$V(u_{\overline{y}}(t)) = c, \quad \text{for all } t \in J_{\overline{y}}. \tag{4.171}$$

Thus, differentiating on both sides of (4.171) with respect to t

$$\dot{V}(u_{\overline{y}}(t)) = 0, \quad \text{for all } t \in J_{\overline{y}},$$

so that, in particular,

$$\dot{V}(\overline{y}) = 0,$$

which shows that $\overline{y} \in S$, where S is as defined in (4.159). We have therefore shown that

$$\omega(\gamma_p) \subseteq S;$$

consequently, since $\omega(\gamma_p)$ is invariant, by Proposition 4.6.3, and M is the largest invariant set contained in S, it follows that

$$\omega(\gamma_p) \subseteq M,$$

which was to be shown. ■

Example 4.9.16 (Negative Gradient Flow Revisited)**.** Let U be an open subset of $\mathbb{R}^N$ and let $V\colon U \to \mathbb{R}$ be a C^2 function. Put $F(x) = -\nabla V(x)$ for all $x \in U$. Assume that V has a (strict) local minimum at $\overline{x} \in U$; that is, there exists $r > 0$ such that $\overline{B_r(\overline{x})} \subset U$ and

$$V(\overline{x}) < V(y), \qquad \text{for all } y \in \overline{B_r(\overline{x})} \backslash \{\overline{x}\}. \tag{4.172}$$

Assume also that $\overline{B_r(\overline{x})} \backslash \{\overline{x}\}$ contains no equilibrium points of F.

We show that there exists an open neighborhood, Ω, of $\overline{x}$, such that $\overline{\Omega} \subset U$, with the property that

$$p \in \Omega \Rightarrow \omega(\gamma_p) = \{\overline{x}\}. \tag{4.173}$$

In order to prove this claim, we let

$$\varepsilon = \min_{x \in \partial B_{B_r(\overline{x})}} (V(x) - V(\overline{x})). \tag{4.174}$$

It follows from (4.172), the continuity of V, and the compactness of $\partial B_{B_r(\overline{x})}$ that $\varepsilon > 0$.

Put

$$\Omega = \{x \in U \mid V(x) < V(\overline{x}) + \varepsilon\}, \tag{4.175}$$

where ε is as defined in (4.174). Then, Ω is an open neighborhood of $\overline{x}$.

Observe that V is a Liapunov function in U. In fact, using the assumption that $F = -\nabla V$, we get that

$$\dot{V}(x) = -\|\nabla V(x)\|^2, \qquad \text{for all } x \in U, \tag{4.176}$$

where we have used (4.157) and the assumption that $F = -\nabla F$, so that

$$\dot{V}(x) \leqslant 0, \qquad \text{for all } x \in U. \tag{4.177}$$

Let $p \in \Omega$, where Ω is as defined in (4.175). It then follows that

$$V(p) < V(\overline{x}) + \varepsilon. \tag{4.178}$$

Let $u_p\colon J_p \to U$ denote the unique solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} &= F(x); \\[2ex] x(0) &= p, \end{cases} \tag{4.179}$$

where J_p is the maximal interval of existence.

Next, apply the Chain Rule and use the definition of $\dot{V}$ in (4.157) to obtain that

$$\frac{d}{dt}\left[V(u_p(t))\right] = \dot{V}(u_p(t)), \qquad \text{for all } t \in J_p,$$

so that

$$\frac{d}{dt}\left[V(u_p(t))\right] = -\|\nabla V(u_p(t))\|, \qquad \text{for all } t \in J_p, \tag{4.180}$$

by virtue of (4.176). It follows from (4.180) that

$$\frac{d}{dt}\left[V(u_p(t))\right] \leqslant 0, \quad \text{for all } t \in J_p, \tag{4.181}$$

from which we get that

$$V(u_p(t)) \leqslant V(p), \quad \text{for all } t \in J_p \cap [0, \infty). \tag{4.182}$$

Combining (4.178) and (4.182) yields that

$$V(u_p(t)) < V(\overline{x}) + \varepsilon, \quad \text{for all } t \in J_p \cap [0, \infty),$$

which shows that

$$\gamma_p^+ \subset \Omega. \tag{4.183}$$

Next, we show that

$$u_p(t) \in B_r(\overline{x}), \quad \text{for all } t \in J_p \cap [0, \infty). \tag{4.184}$$

If not, by continuity and the intermediate value theorem, there exists $t_1 > 0$ such that

$$\|u_p(t_1)\| = r.$$

It then follows from (4.174) that

$$V(u_p(t_1)) \geqslant V(\overline{x}) + \varepsilon. \tag{4.185}$$

On the other hand, it follows from (4.181) and (4.178) that

$$V(u_p(t_1)) \leqslant V(p) < V(\overline{x}) + \varepsilon,$$

which is in direct contradiction with (4.185). We have therefore established (4.184).

Finally, the assumption that $\overline{B_r(\overline{x})}\backslash\{\overline{x}\}$ contains no equilibrium points of $F = -\nabla V$ implies that

$$S = \{x \in \Omega \mid \dot{V}(x) = 0\} = \{\overline{x}\};$$

consequently,

$$M = \{\overline{x}\}. \tag{4.186}$$

In view of (4.177), (4.183) and (4.184), we can now apply Theorem 4.9.15 to obtain (4.173) from (4.186), which we wanted to show.

Chapter 5

Two–Dimensional Systems

The mail goal of this chapter is to give an analysis of the system

$$\begin{cases} \dfrac{dx}{dt} &= y - \mu \left(\dfrac{x^3}{3} - x \right); \\ \\ \dfrac{dy}{dt} &= -x, \end{cases} \tag{5.1}$$

where μ is a real parameter. The system in (5.1) is known as the Lienard equations and comes about as a way to solve the second order, nonlinear differential equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{5.2}$$

where

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{d^2x}{dt^2}.$$

The equation in (5.2) is known as the Van der Pol equation in electric circuits theory. Notice that if $(x(t), y(t))$ is a solution of the system in (5.1) then $x = x(t)$ is a solution to the second order differential equation in (5.2). In fact, differentiating the first equation in (5.1) with respect to t we obtain,

$$\ddot{x} = \dot{y} - \mu(x^2 - 1)\dot{x}. \tag{5.3}$$

Substituting the second equation in (5.1) into (5.3) leads to (5.2).

In the next section we will consider the case $\mu < 0$ and use the method of Liapunov discussed in the previous section to show that $(0, 0)$ is asymptotically stable. More precisely, there exists a neighborhood, Ω, of the origin, $(0, 0)$, such that, if $(p, q) \in \Omega$, the $\omega(\gamma_{(p,q)}) = \{(0, 0)\}$. In a subsequent section, we consider the question what the α–limit set of $\gamma_{(p,q)}$ would be. This is equivalent to looking $u_{(p,q)}(-t) = (x(-t), y(-t))$ as $t \to \infty$. Set $z(t) = x(-t)$ for all $t \in \mathbb{R}$ such that $-t \in J_{(p,q)}$. Then,

$$\dot{z}(t) = -\dot{x}(-t) \quad \text{and} \quad \ddot{z}(t) = \ddot{x}(-t),$$

for all $t \in \mathbb{R}$ such that $-t \in J_{(p,q)}$. It then follows from (5.2) that z solves the second order equation

$$\ddot{z} - \mu(x^2 - 1)\dot{z} + z = 0. \tag{5.4}$$

The Lienard system corresponding to (5.4) is then

$$\begin{cases} \dfrac{dx}{dt} & = & y - \nu\left(\dfrac{x^3}{3} - x\right); \\ \\ \dfrac{dy}{dt} & = & -x, \end{cases} \tag{5.5}$$

where $\nu = -\mu > 0$ for the case $\mu < 0$. In Section 5.2 we will show that that the system in (5.5) has a unique asymptotically stable limit cycle for $\nu > 0$. Thus, the Van del Pol equation in (5.2) has a non–trivial periodic solution.

5.1 Analysis of the Lienard System: Part I

Consider the Lienard system in (5.1),

$$\begin{cases} \dfrac{dx}{dt} & = & y - \mu\left(\dfrac{x^3}{3} - x\right); \\ \\ \dfrac{dy}{dt} & = & -x, \end{cases} \tag{5.6}$$

where $\mu < 0$. Observe that $(0, 0)$ is the only equilibrium point of the system. We will apply Theorem 4.9.15 on page 81 to prove that there exists a neighborhood, Ω, of $(0, 0)$ in $\mathbb{R}^2$ with the property that

$$(p, q) \in \Omega \Rightarrow \omega(\gamma_{(p,q)}) = \{(0, 0)\}. \tag{5.7}$$

In order to prove (5.8), define $V : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$V(x, y) = x^2 + y^2, \quad \text{for all } (x, y) \in \mathbb{R}^2, \tag{5.8}$$

Next, compute

$$\begin{aligned} \dot{V}(x, y) &= 2x\left(y - \mu\left(\frac{x^3}{3} - x\right)\right) - 2xy \\ &= -2\mu x^2\left(\frac{x^2}{3} - 1\right), \end{aligned} \tag{5.9}$$

so that

$$\dot{V}(x, y) \leqslant 0, \quad \text{provided that } x^2 \leqslant 3, \tag{5.10}$$

since we are assuming that $\mu < 0$. Define

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid V(x, y) < 3\}. \tag{5.11}$$

It then follows from (5.9) that V is a Liapunov function in Ω.

Next, we show that

$$\omega(\gamma_{(p,q)}) = \{(0,0)\}, \quad \text{for all } (p,q) \in \Omega. \tag{5.12}$$

In order to prove (5.12), we first verify the hypotheses of the S&M Theorem (Theorem 4.9.15 on page 81) for the semi–orbit $\gamma^+_{(p,q)}$ and the set Ω defined in (5.11). Since Ω is bounded, it suffices to show that

$$\gamma^+_{(p,q)} \subset \Omega, \quad \text{for all } (p,q) \in \Omega. \tag{5.13}$$

Assume that $(p,q) \in \Omega$. Then,

$$V(p,q) < 3. \tag{5.14}$$

We show that

$$V(u_{(p,q)}(t)) < 3, \quad \text{for all } t > 0. \tag{5.15}$$

If (5.15) not the case, there exists $t_1 > 0$ such that

$$V(u_{(p,q)}(t_1)) = 3, \tag{5.16}$$

and

$$V(u_{(p,q)}(t)) < 3, \quad \text{for } 0 \leqslant t < t_1. \tag{5.17}$$

It follows from (5.17) and (5.10) that

$$\dot{V}(u_{(p,q)}(t)) \leqslant 0, \quad \text{for } 0 < t < t_1,$$

from which we get that

$$V(u_{(p,q)}(t)) \leqslant V(p,q) < 3, \quad \text{for } 0 \leqslant t < t_1, \tag{5.18}$$

where we have also used (5.14). I follows from (5.19) and continuity that

$$V(u_{(p,q)}(t_1)) < 3, \tag{5.19}$$

which is in direct contradiction with (5.16). This contradiction establishes (5.15), which shows that (5.13) holds true. We can therefore apply the S&M Theorem to conclude that

$$\omega(\gamma_{(p,q)}) \subseteq M, \quad \text{for all } (p,q) \in \Omega, \tag{5.20}$$

where M is the largest invariant subset of

$$S = \{(x,y) \in \Omega \mid \dot{V}(x,y) = 0\}. \tag{5.21}$$

It follows from (5.8), (5.9) and (5.21) that

$$S = \{(0,y) \in \Omega \mid -\sqrt{3} < y < \sqrt{3}\}. \tag{5.22}$$

Next, let $(0, y_o) \in S$ be such that $y_o > 0$ and write $u_{(0,y_o)}(t) = (x(t), y(t))$ for all $t \in J_{(0,y_o)}$. It follows from the first equation in (5.6) that

$$x'(o) = y_o > 0.$$

Consequently, by the continuity of the derivative of $u_{(0,y_o)}$, there exists $\delta > 0$ such that $[0, \delta) \in J_{(0,y_o)}$ and

$$x'(t) > \frac{y_o}{2}, \quad \text{for all } t \in [0, \delta). \tag{5.23}$$

It then follows from (5.23) that

$$x(t) - x(0) = \int_0^t x'(\tau)\ d\tau > \frac{y_o}{2}\ t, \quad \text{for all } t \in (0, \delta),$$

from which we get that

$$x(t) > 0, \quad \text{for all } t \in (0, \delta),$$

and so the orbit $\gamma_{(0,y_o)}$, for $y_o > 0$ must leave the set S given in (5.22). Similarly, the orbit $\gamma_{(0,y_o)}$, for $y_o < 0$ must leave the set S. Hence, the largest invariant set, M, contained in S must be $\{(0,0)\}$; in other words, $M = \{(0,0)\}$. Hence, it follows from (5.20) that

$$\omega(\gamma_{(p,q)}) = \{(0,0)\}, \quad \text{for all } (p,q) \in \Omega,$$

which was to be shown.

5.2 Analysis of the Lienard System: Part II

Appendix A

Definitions and Facts from Real Analysis

In this Appendix, we list all the concepts and results from analysis that are used in these notes. They may be found in undergraduate texts on real analysis (see, for example, Bartle's *The Elements of Real Analysis,* [Bar76]). Another good reference is the first three chapters of Spivak's *Calculus on Manifolds,* [Spi65].

A.1 Topology of Euclidean Space

The open ball of radius r about p in N–dimensional Euclidean space, $\mathbb{R}^N$, is the set

$$B_r(p) = \{x \in \mathbb{R}^N \mid \|x - p\| < r\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$; in other words,

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}, \quad \text{for all } x \in \mathbb{R}^N.$$

The closure of $B_r(p)$, denoted by $\overline{B}_r(p)$, is defined by

$$\overline{B}_r(p) = \{x \in \mathbb{R}^N \mid \|x - p\| \leqslant r\}.$$

Remark A.1.1. In the one–dimensional case, $N = 1$, the open ball, $B_r(p)$, for $p \in \mathbb{R}$, will simply be the open interval $(p - r, p + r)$.

Definition A.1.2 (Open Sets)**.** A subset, U, of $\mathbb{R}^N$ is said to be open if, for every $p \in U$, there exists $r > 0$ such that

$$B_r(p) \subset U.$$

Definition A.1.3 (Closed Sets)**.** A subset, K, of $\mathbb{R}^N$ is said to be closed if its complement,

$$K^c = \{x \in \mathbb{R}^N \mid x \notin K\}$$

is an open subset of $\mathbb{R}^N$.

Proposition A.1.4 (Facts About Open Sets)**.** Let U, V, U_α, for α in some indexing set Λ, denote open subsets of $\mathbb{R}^n$. Then,

1. the finite intersection, $U \cap V$, is open;
2. the (possibly infinite) union $\displaystyle\bigcup_{\alpha\in\Lambda} U_\alpha$ is open;
3. U^c is closed.

Proposition A.1.5 (Facts About Closed Sets)**.** Let K, L, K_α, for α in some indexing set Λ, denote closed subsets of $\mathbb{R}^n$. Then,

1. the finite union, $K \cup L$, is closed;
2. the (possibly infinite) intersection $\displaystyle\bigcap_{\alpha\in\Lambda} K_\alpha$ is closed;
3. K^c is open.

Definition A.1.6 (Relative Topology)**.** Let X be subset of $\mathbb{R}^N$. A subset, Y, of X, is set to be relatively open in X if there exists an open set $U \subseteq \mathbb{R}^N$ such that $Y = U \cap X$. Similarly, Y is relatively closed in X if there exists a closed set $K \subseteq \mathbb{R}^N$, such that $Y = K \cap X$. Observe that the empty set, $\emptyset$, and X are relatively open and closed in X.

Definition A.1.7 (Connectedness)**.** A subset X of $\mathbb{R}^N$ is said to be connected if the only subsets of X which are both relatively open and relatively closed in X are $\emptyset$ and X.

Example A.1.8 (Intervals in $\mathbb{R}$)**.** Intervals of real numbers are connected in $\mathbb{R}$.

Definition A.1.9 (Open Cover)**.** Let X be subset of $\mathbb{R}^N$. A collection of open sets, $\{U_\alpha \mid \alpha \in \Lambda\}$, for some indexing set Λ, is said to be an open cover for X if and only if

$$X \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$$

Definition A.1.10 (Compactness)**.** A subset, K, of $\mathbb{R}^N$ is said to be compact if and only if every open cover for K has a finite subcover; that is, for every collection of open set, $\{U_\alpha \mid \alpha \in \Lambda\}$, such that

$$X \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha,$$

there exist finitely many indices, $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that

$$X \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}.$$

Example A.1.11 (Examples of Compact Sets)**.** Let a, b and R denote real numbers with $a < b$ and $R > 0$.

1. The closed and bounded interval, $[a, b]$, is compact.
2. Closed and bounded subsets of $\mathbb{R}^N$ are compact; where a set $X \subset \mathbb{R}^N$ is said to be bounded if there exists $R > 0$ such that $X \subset B_R(0)$.
3. Finite sets of points are compact.

Proposition A.1.12. Let U be an open subset of $\mathbb{R}^N$ and K a compact subset, K, of U. There exists and open subset, V, of U such that

$$K \subset V \subset \overline{V} \subset U,$$

where $\overline{V}$ is compact.

Proof: Let U be an open subset of $\mathbb{R}^N$ and and $K \subset U$ be compact. Then, for each $x \in K$ there exists a positive real number r_x such that

$$B_{2r_x}(x) \subset U.$$

Consequently,

$$\overline{B_{r_x}(x)} \subset U, \qquad \text{for all } x \in K.$$

Since K is compact, the open cover, $\{B_{r_x}(x)\}_{x \in K}$, for K has a finite subcover,

$$B_{r_1}(x_1), B_{r_2}(x_2), \ldots, B_{r_n}(x_n).$$

We then have that

$$K \subset \bigcup_{i=1}^{n} B_{r_1}(x_1).$$

Put

$$V = \bigcup_{i=1}^{n} B_{r_i}(x_i).$$

Then V is open with $K \subset V$. Furthermore,

$$\overline{V} \subset \bigcup_{i=1}^{n} \overline{B_{r_i}(x_i)} \subset \bigcup_{i=1}^{n} B_{2r_i}(x_i) \subset U.$$

Observe that the set $\overline{V}$ is bounded. In fact, from

$$\overline{V} \subset \bigcup_{i=1}^{n} B_{2r_i}(x_i),$$

we obtain that

$$\overline{V} \subset B_{R}(0),$$

where

$$R = \max_{1 \leqslant i \leqslant n} \|x_i\| + 2 \max_{1 \leqslant i \leqslant n} r_i.$$

It then follows that $\overline{V}$ is compact. ■

Proposition A.1.13. Let K denote a compact subset of $\mathbb{R}^N$, and F a closed subset of $\mathbb{R}^N$. If $K \cap F = \emptyset$, then $\text{dist}(K, F) > 0$; where

$$\text{dist}(K, F) = \inf_{x \in K,\ y \in F} \|x - y\|.$$

A.2 Sequences in Euclidean Space

Definition A.2.1 (Convergence)**.** A sequence of points, (x_m), in $\mathbb{R}^N$ is said to converge to $x \in \mathbb{R}^N$ if

$$\lim_{m\to\infty} \|x_m - x\| = 0.$$

Definition A.2.2 (Cauchy Sequence)**.** A sequence of points, (x_m), in $\mathbb{R}^N$ is said to be a Cauchy sequence if, for every $\varepsilon > 0$, there exists $M \in \mathbf{N}$ such that

$$m, n \geqslant M \Rightarrow \|x_m - x_n\| < \varepsilon.$$

Proposition A.2.3 (Completeness)**.** Evert Cauchy sequence in $\mathbb{R}^N$ converges; that is, if (x_m) is a Cauchy sequence in $\mathbb{R}^N$, then there exists $x \in \mathbb{R}^N$ such that

$$\lim_{m\to\infty} \|x_m - x\| = 0.$$

Definition A.2.4 (Subsequences)**.** Let (x_m) be a sequence of points in $\mathbb{R}^N$ and (m_k) an infinite sequence of natural numbers satisfying $m_k < m_{k+1}$ for all $k \in \mathbf{N}$. Then, the sequence (x_{m_k}) is called a subsequence of (x_m).

Proposition A.2.5 (Bolzano–Weierstrass Property of Compact Sets)**.** A subset, K, of $\mathbb{R}^N$ is compact if and only if every infinite subset of K has a subsequence which converges to some point in K.

A.3 Properties of Continuous Functions

Definition A.3.1 (Continuity at a Point)**.** *Let U be an open subset of $\mathbb{R}^N$. A function*

$$F : U \to \mathbb{R}^m$$

is said to be continuous at $x \in U$ if and only if

$$\lim_{\|y-x\|\to 0} \|F(y) - F(x)\| = 0.$$

Definition A.3.2 (Continuity on a Set)**.** *Let X be a subset of $\mathbb{R}^N$. A function*

$$F : X \to \mathbb{R}^m$$

is said to be continuous on X if and only if the pre-image of every open subset, V, of $\mathbb{R}^m$,

$$F^{-1}(V) = \{x \in X \mid F(x) \in V\},$$

is relatively open in X.

Theorem A.3.3 (Intermediate Value Theorem)**.** Let X be a subset of $\mathbb{R}^N$ and I and open interval of real numbers,

1. If $F\colon X \to \mathbb{R}^m$ is continuous and X is connected, then image of X under F,
$$F(X) = \{y \in \mathbb{R}^m \mid y = f(x), \text{ for some } x \in X\},$$
is a connected subset of $\mathbb{R}^m$.

2. Suppose that $f\colon I \to \mathbb{R}$ is continuous, and let $a, b \in I$ be such that $a < b$. If
$$f(a) < y < f(b)$$
or
$$f(a) > y > f(b),$$
for some $y \in \mathbb{R}$, then there exists $x \in (a, b)$ such that
$$f(x) = y.$$

Theorem A.3.4 (Extremal Value Theorem)**.** Let K be a compact subset of $\mathbb{R}^N$.

1. If $F\colon K \to \mathbb{R}^m$ is continuous, then the image of K under F, $F(X)$, is compact.

2. Suppose that $K \neq \emptyset$. Let $f\colon K \to \mathbb{R}$ be continuous. Then, there exist x_1 and x_2 in K such that
$$F(x_1) = \max_{x \in K} f(x)$$
and
$$F(x_2) = \min_{x \in K} f(x)$$

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