Assignment #3

Due on Friday, February 3, 2012

Read Section 2.3.1 on *Nondimensionalization,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Do the following problems.

1. Nondimensionalize the Logistic growth equation for bacterial growth in a medium with carrying capacity K: $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$, where r is the intrinsic growth rate of the population, by introducing dimensionless variables

$$u = \frac{N}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda}.$$

   Give interpretations for the scaling parameters μ and λ.

2. Consider again the chemostat model without flow in or out of a single chamber depicted in Figure 1. Proceed as in Problems 1–4 in Assignment 1 assuming this

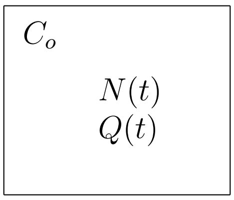

Figure 1: One–Compartment Chemostat Model

   time that the *per capita* growth rate is given by the Michaelis-Menten enzyme kinetics relation

$$K(c) = \frac{rc}{a + c},$$

   where $c = Q/V$ is the nutrient concentration in the growth medium, to derive a differential equations for the bacterial density, $n = N/V$, and the nutrient concentration. You will need to use the yield $Y = 1/\alpha$, or the number of new cells produced in the chemostat due to consumption of one unit of nutrient.

   Give an interpretation for the parameter r.

3. The differential equation
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - p(N, \Lambda), \tag{1}$$
models a population that is subject to predation reflected in the term $p(N, \Lambda)$, which depends on the population size, N, and a set of parameters, Λ. In the absence of predation the population undergoes logistic growth with intrinsic growth rate, r, and carrying capacity, K.

   In 1978, Ludwig, Jones and Holling published an article in the Journal of Animal Ecology (*Qualitative analysis of insect outbreak systems: the spruce budworm and forest*, Volume 47, pp. 315–332) in which they proposed the following constitutive equation for the predation term,
$$p(N, a, b) = \frac{bN^2}{a^2 + N^2}. \tag{2}$$

   (a) Give interpretations for the parameters a and b in (2).

   (b) Nondimensionalize the differential equation in (1) by introducing dimensionless variables
$$u = \frac{N}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda},$$
   to obtain the dimensionless equation
$$\frac{du}{d\tau} = \alpha u\left(1 - \frac{u}{\beta}\right) - \frac{u^2}{1 + u^2}, \tag{3}$$
   where α and β are dimensionless parameters.

   Express α and β in terms of the parameters r, K, a and b.

4. Observe that $u = 0$ is an equilibrium point of the equation in (3). Determine the nature of the stability of this equilibrium point.

5. The equation
$$\alpha\left(1 - \frac{u}{\beta}\right) - \frac{u}{1 + u^2} = 0, \tag{4}$$
which yields the non–zero equilibrium points of (3), cannot be easily solved algebraically.

   (a) Explain why the equation in (4) must have at least one real solution, and at most three distinct real solutions.

   (b) Determine conditions on α and β that will guarantee that the equation in (4) will have (i) exactly one real solution, (ii) two distinct real solutions, and (iii) three distinct real solutions.