Assignment #7

Due on Wednesday, March 7, 2012

Read Section 4.1 on *Random Variables* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.

Do the following problems.

1. Recall that two events, A and B, are said to be stochastically independent if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

Assume that A and B are stochastically independent. Prove that

   (a) A and B^c are stochastically independent;
   (b) A^c and B are stochastically independent; and
   (c) A^c and B^c are stochastically independent.

2. Given a discrete random variable X with a finite number of possible values

$$x_1, x_2, x_3, \ldots, x_N,$$

the expected value of X is defined to be the sum $E(X) = \sum_{i=1}^{N} x_i P[X = x_i]$. Use this formula to compute the expected value of the numbers appearing on the top face of a fair die. Explain the meaning of this number.

3. Three discrete random variables, X_1, X_2 and X_3, are said to be mutually independent if

$$\Pr(X_i = a, X_j = b) = \Pr(X_i = a) \cdot \Pr(X_j = b), \quad \text{for } i \neq j,$$

for all values of a and b; that is, X_1, X_2 and X_3 are pairwise stochastically independent, and

$$\Pr(X_1 = a, X_2 = b, X_3 = c) = \Pr(X_1 = a) \cdot \Pr(X_2 = b) \cdot \Pr(X_3 = c),$$

for all values of a, b and c. Set $Y = X_1 + X_2$. Prove that Y and X_3 are stochastically independent.

4. Consider a hypothetical experiment in which there are only three bacteria in a culture. Suppose that each bacteirum has a small probability p, with $0 < p < 1$, of developing a mutation in a short time interval. Number the bacteria 1, 2 and 3. Use the symbol M to denote the given bacterium mutates in the short time interval, and N to denote that the bacterium did not mutate in that interval.

   (a) List all possible outcomes of the experiment using the symbols M or N, for each of the bacteria 1, 2 and 3, to denote whether a bacterium mutated or not, respectively. This will generate triples made up of the symbols M and N. What is the probability of each outcome?

   (b) Let Y denote the number of bacteria that mutate in the short time interval. This defines a discrete random variable. List the possible values for Y and give the probability for each of these values. In other words, give the probability mass function for Y.

   (c) Compute the expected value of Y.

5. Repeat the procedure in Problem 4 in the case of four bacteria, each having a probability p of mutating in a short time interval.

   Generalize to the case of N bacteria, each having a probability p of mutating in a short time interval.

   For this part of the problem it will be helpful to know that the number of different ways of choosing m bacteria out of N is given by the combinatorial expression

   $$\binom{N}{m} = \frac{N!}{m!(N-m)!},$$

   for $m = 0, 1, 2, \ldots, N$. The symbol $\binom{N}{m}$ is read "N choose m."

   Note: The distribution for Y obtained in this problem is called the *binomial* distribution with parameters N and p. We write $Y \sim \text{Binomial}(N, p)$.