Solutions to Review Problems for Exam #1

1. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 1. The

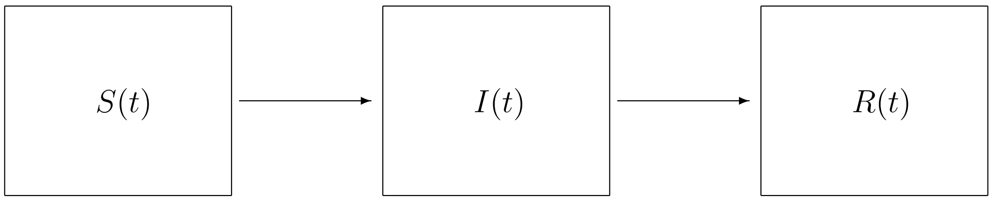

Figure 1: SIR Compartments

first compartment, $S(t)$, denotes the set of individuals in a population that are susceptible to acquiring the disease; the second compartment, $I(t)$, denotes the set of infected individual who can also infect others; and the third compartment, $R(t)$, denotes the set of individuals who had the disease and who have recovered from it; they can no longer get infected.

Assume that the total number of individuals in the population,

$$N = S(t) + I(t) + R(t),$$

is constant. Susceptible individuals can get infected by contact with infectious individuals and move to the infected class. This is indicated by the arrow going from the $S(t)$ compartment to the $I(t)$ compartment.

The rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is proportional to the number of infected individuals with constant of proportionality $\gamma > 0$. What are the units for β and γ?

Use conservation principles to derive a system of differential equations for the functions S, I and R, assuming that they are differentiable. Models of this type were first studied by Kermack and McKendrick in the early 1930s.

Introduce dimensionless variables

$$\widehat{s}(t) = \frac{S(t)}{N}, \quad \widehat{i}(t) = \frac{I(t)}{N}, \quad \widehat{r}(t) = \frac{R(t)}{N}, \quad \text{and} \quad \widehat{t} = \frac{t}{\tau}, \tag{1}$$

for some scaling factor, τ, in units of time, in order to write the system in dimensionless form.

***Solution**:* Using conservation principles on each of the compartments, we obtain the system of ordinary differential equations

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI; \\ \dfrac{dI}{dt} = \beta SI - \gamma I; \\ \dfrac{dR}{dt} = \gamma I. \end{cases} \tag{2}$$

It follows from the equations in (2) that β has units of 1/[time $\times$ individual], while γ has units of 1/time.

Next, use the change of variables in (1) and the Chain Rule to obtain from the first equation in (2) that

$$\begin{aligned} \frac{d\widehat{s}}{d\widehat{t}} &= \frac{d\widehat{s}}{dt}\cdot\frac{dt}{d\widehat{t}} \\ &= \frac{\tau}{N}\frac{dS}{dt} \\ &= -\frac{\tau}{N}\beta SI, \end{aligned}$$

so that, using (1) again,

$$\frac{d\widehat{s}}{d\widehat{t}} = -\beta\tau N\,\widehat{s}\,\widehat{i}. \tag{3}$$

Similar calculations for the second equation in (2) yield

$$\frac{d\widehat{i}}{d\widehat{t}} = \beta\tau N\,\widehat{s}\,\widehat{i} - \gamma\tau\widehat{i}; \tag{4}$$

and, for the third equation in (4),

$$\frac{d\widehat{r}}{d\widehat{t}} = \gamma\tau\widehat{i}. \tag{5}$$

Define the dimensionless parameter

$$\beta\tau N = R_o, \tag{6}$$

and set

$$\gamma\tau = 1,$$

so tat

$$\tau = \frac{1}{\gamma}, \tag{7}$$

and

$$R_o = \frac{\beta N}{\gamma}, \tag{8}$$

by virtue of (6).

Next, substitute (6) and (7) into the equations in (3), (4) and (5) to obtain the dimensionless system

$$\begin{cases} \dfrac{d\widehat{s}}{d\widehat{t}} &= -R_o\, \widehat{s}\, \widehat{i}; \\[2ex] \dfrac{d\widehat{i}}{d\widehat{t}} &= R_o\, \widehat{s}\, \widehat{i} - \widehat{i}; \\[2ex] \dfrac{d\widehat{r}}{d\widehat{t}} &= \widehat{i}. \end{cases} \tag{9}$$

If we stipulate from the outset that t is measured in units of $1/\gamma$ and s, i and r are measures in fractions of the total population, N, then the system in (9) can be written in simpler form as

$$\begin{cases} \dfrac{ds}{dt} &= -R_o si; \\[2ex] \dfrac{di}{dt} &= R_o si - i; \\[2ex] \dfrac{dr}{dt} &= i, \end{cases}$$

which depends on the single dimensionless parameter, R_o, given in (8). □

2. **Modeling Traffic Flow.** Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} &= 0; \\[2ex] u(x,0) &= f(x), \end{cases} \tag{10}$$

where

$$g(u) = u(1-u), \tag{11}$$

and the initial condition f is given by

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \frac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1. \end{cases} \tag{12}$$

(a) Sketch the characteristic curves of the partial differential equation.

***Solution**:* The equation for the characteristic curves is given by

$$\frac{dx}{dt} = g'(u). \tag{13}$$

On characteristic curves, a solution, u, to the partial differential equation in (10) satisfies the ordinary differential equation

$$\frac{du}{dt} = 0,$$

which shows that u is constant along characteristic curves. We write

$$u(x,t) = \varphi(k), \tag{14}$$

where $\varphi(k)$ is the constant value of u on the characteristic indexed by k.

Using the value for u in (14), the equation for the characteristic curves in (13) can be re–written as

$$\frac{dx}{dt} = g'(\varphi(k)). \tag{15}$$

Solving the differential equation in (15) yields the equation for the characteristic curves

$$x = g'(\varphi(k))t + k, \tag{16}$$

where the parameter k corresponds to the value on the x–axis on which the characteristic curves meet the x–axis.

Next, solve for k in (16) and substitute into (14) to obtain the expression

$$u(x,t) = \varphi(x - g'(u(x,t))t), \tag{17}$$

which gives a solution of the partial differential equation in (10) implicitly.

Using the initial condition in (10), we obtain from (17) that

$$\varphi(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that (17) can now be re–written as

$$u(x,t) = f(x - g'(u(x,t))t). \tag{18}$$

Accordingly, the equation for the characteristic curves in (16) can now be re–written as

$$x = g'(f(k))t + k, \tag{19}$$

so that the characteristic curves will be straight lines in the xt–plane of slope $1/g'(f(k))$ going through $(k,0)$ for $k \in \mathbb{R}$, where $g'(u)$ is obtained from (11) as

$$g'(u) = 1 - 2u. \tag{20}$$

For instance, using (20), (12) and (19) we get that the equations for the characteristic curves for $k \leqslant -1$ are given by

$$x = -t + k, \quad \text{for } k \leqslant -1. \tag{21}$$

The curves described by (21) are straight lines with slope -1 going through $(k,0)$, for $k \leqslant -1$. Some of these are pictured in Figure 2. Similarly, for

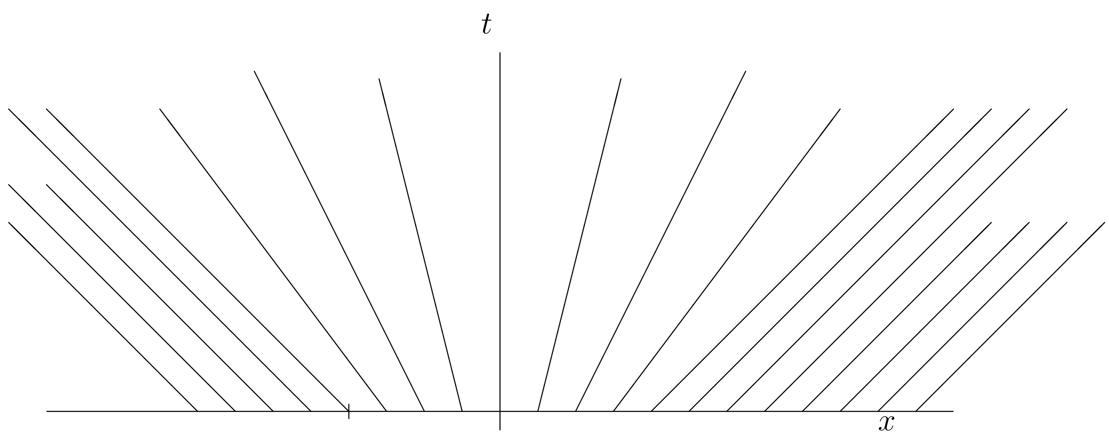

Figure 2: Characteristic Curves for Problem (10)

$k \geqslant 1$, the curves in (19) have equations

$$x = t + k, \quad \text{for } k \geqslant 1,$$

which are straight lines of slope 1 going through $(k, 0)$, for $k \geqslant 1$; some of these lines are also sketched in Figure 2.

For values of k between -1 and 1, the slopes of the lines in (19) are given by $1/g'(f(k))$, where $f(k)$ ranges from 1 at $k = -1$, to 0 at $k = 1$; so, according to (20), the slopes of the lines are negative and increase in absolute value to $infty$ as k approaches 0. At $k = 0$, $f(k) = 1/2$, so that $g'(f(k)) = 0$, by virtue of (20), so that the characteristic curve will be $x = 0$, according to (19), or the t–axis. As k ranges from 0 to 1, the characteristic curves fan out from the t–axis to the line $x = t + 1$. A few of these curves are shown in Figure 2. □

(b) Explain how the initial value problem can be solved in this case, and give a formula for $u(x, t)$.

***Solution**:* Since the characteristic curves do not intersect for $t > 0$, the initial value problem in (10) can always be solved by traveling back along the characteristic curves until the hit the x–axis at a point $(k, 0)$, and then reading the value of the initial density, $u(k, 0) = f(k)$, at that point. For example, if the point (x, t) lies in the region $x < -t - 1$, we see from Figure 2 that the characteristic curve containing the point (x, t) will meet the x–axis at some point $(k, 0)$ with $k < -1$; since, $f(k) = 1$ for $k < -1$, it follows from (18) that

$$u(x, t) = 1, \quad \text{for } x < -t - 1, \text{ and } t \geqslant 0. \tag{22}$$

Similarly, if $x \geqslant x + t$, then the characteristic curve containing (x, t) will meet the x–axis at some point $(k, 0)$ with $k \geqslant 1$; since $f(k) = 0$ for $k \geqslant 1$, it follows from (18) that

$$u(x, t) = 0, \quad \text{for } x \geqslant t + 1, \text{ and } t \geqslant 0. \tag{23}$$

For (x, t) lying in the region between the lines $x = -t-1$ and $x = t+1$, the characteristic curve containing the point will meet the x–axis at a point $(k, 0)$ with $-1 \leqslant k \leqslant 1$. Since $f(k) = \dfrac{1}{2}(1 - k)$ for those values of k, by (12), it follows from (18) that

$$u(x, t) = \frac{1}{2}[1 - (x - g'(u(x, t))t)], \quad \text{for } -t - 1 \leqslant x \leqslant t + 1. \tag{24}$$

Using (20), we can re–write (24) as

$$u(x, t) = \frac{1 - x + t}{2} - u(x, t)t, \quad \text{for } -t - 1 \leqslant x \leqslant t + 1. \tag{25}$$

Solving for $u(x,t)$ in (25) yields

$$u(x,t) = \frac{1-x+t}{2(1+t)}, \quad \text{for } -t-1 \leqslant x \leqslant t+1. \tag{26}$$

Finally, putting together the results in (22), (23) and (26), we obtain the following formula for $u(x,t)$:

$$u(x,t) = \begin{cases} 1, & \text{for } x < -t-1; \\ \dfrac{1-x+t}{2(1+t)}, & \text{for } -t-1 \leqslant x \leqslant t+1; \\ 0, & \text{for } x > t+1, \end{cases}$$

for $t \geqslant 0$. □

3. **Age Structured Population Models.** Postulate a population density, $n(a,t)$, which also gives the age distribution for individuals in the population; so that, the number of individuals in the population between the ages a_1 and a_2 at time t is given by $\displaystyle\int_{a_1}^{a_2} n(a,t)\; da$.

    (a) Explain why $n(a,t)$ is given in units of population divided by units of time.
    ***Solution**:* Since $n(a,t)\Delta a$ gives, approximately, the number of individuals in the population with ages between a and $a+\Delta a$, and a is measured in chronological time, it follows that the units of n are individuals in the population per unit time. □

    (b) Since a is a function of t, assuming that n is C^1, we can use Chain Rule to compute the rate of change of population density at time t, $\dfrac{dn}{dt}$.
    Explain why

    $$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial a}. \tag{27}$$

    ***Solution**:* Applying the Chain Rule we obtain

    $$\frac{dn}{dt} = \frac{\partial n}{\partial t}\cdot\frac{dt}{dt} + \frac{\partial n}{\partial a}\cdot\frac{da}{dt}. \tag{28}$$

    Since the age, a, of individuals in the population is measured in chronological time, it follows that

    $$\frac{da}{dt} = 1. \tag{29}$$

    The equation in (27) follows from (28) and (29). □

(c) Assume that death rate for individuals of age a in the population is proportional to the number of individuals at that age with constant of proportionality $\mu(a)$.

Use a conservation principle to derive the following partial differential equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(a)n \tag{30}$$

Give the characteristic curves for the equation.

Solution: At any given age, a, the conservation principle implies that

$$\frac{dn}{dt} = \text{Rate of } n \text{ in } - \text{Rate of } n \text{ out.} \tag{31}$$

Since contributions from births only occur at age $a = 0$, we have that, for $a > 0$,

$$\text{Rate of } n \text{ in } = 0, \tag{32}$$

and

$$\text{Rate of } n \text{ out } = \mu(a)n. \tag{33}$$

Combining the equations (31), (32) and (33) yields the partial differential equation in (30).

The equation for the characteristic curves of (30) is

$$\frac{da}{dt} = 1. \tag{34}$$

Solving the differential equation in (34) yields the equation for the characteristic curves,

$$a = t + k. \tag{35}$$

Thus, the characteristic curves are straight lines of slope 1. □

(d) Give solutions to the partial differential equation derived in the previous part assuming that the death rate is zero for all ages. Interpret your result.

Solution: Assuming that $\mu(a) = 0$, the differential equation in (30)

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = 0. \tag{36}$$

Then, along characteristic curves, n satisfies the ordinary differential equation

$$\frac{dn}{dt} = 0. \tag{37}$$

It follows from (37) that n is constant along characteristic curves, so that

$$n(a,t) = \varphi(k), \tag{38}$$

where $\varphi(k)$ is the constant value of n along the characteristic curve in (35) indexed by k.

Solving for k in (35) and substituting into (38) yields

$$n(a,t) = \varphi(a-t),$$

so that solutions to (36) are traveling waves with speed 1. The initial population distribution simply moves forward in time. □