Review Problems for Exam 2

1. **Poisson Processes and Random Mutations**. It was shown in class and in the lecture notes that, if $M(t)$ denotes the number of mutations that occur in a bacterial colony in the time interval $[0, t]$, then $M(t)$ can be modeled by a Poisson process; in other words, for each $t > 0$, $M(t)$ is a modeled by a Poisson random variable with parameter λt, where the parameter λ denotes the (constant) average number of mutations per unit time. Hence,

$$\Pr[M(t) = m] = \begin{cases} \dfrac{(\lambda t)^m}{m!}\, e^{-\lambda t}, & \text{for } m = 0, 1, 2, 3, \ldots \text{ and } t \geqslant 0; \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Let T_1 denote the time of occurrence of the first mutation. Give the probability density function for T_1 and compute its expected value.

   (b) Compute the limits $\displaystyle\lim_{t\to 0} \frac{\Pr[M(t) = 1]}{t}$ and $\displaystyle\lim_{t\to 0} \frac{\Pr[M(t) \geqslant 2]}{t}$ and give interpretations to your results.

   (c) For each real pair of real numbers, t_1 and t_2, with $t_1 < t_2$, define $Y = M(t_2) - M(t_1)$. Compute the expected value, $E(Y)$, of Y, and give and interpretation for your result.

2. **Random Walk on the Integers**. A particle starts at $x = 0$ and, after one unit of time, it moves one unit to the right with probability p, for $0 < p < 1$, or to the left with probability $1 - p$. Assume that at each time step, whether a particle will move to the right or to the left is independent of where it has been.

   (a) Let X_1 denote the position of the particle after one unit of time and X_2 denote that after 2 units of time. Give the probability distributions for X_1 and X_2 and compute their expectations and variances.

   (b) Let X_3 denote the position of the particle in the previous part after 3 units of time. Give probability distribution, expectation and variance of X_3. Generalize this result to X_n, the position of the particle after n units of time. The set of random variables $\{X_n \mid n = 0, 1, 2, 3, \ldots\}$ is an example of a discrete–time random process

3. **Exponential Distributions**. A continuous random variable, T, is said to have and exponential distribution with parameter $\beta > 0$, if its probability density function, $f_{_T}$, is given by

$$f_{_T}(t) \begin{cases} \dfrac{1}{\beta}\, e^{-t/\beta} & \text{for } t \geqslant 0; \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Compute the conditional probability

   $$\Pr(T > t + s \mid T > t)$$

   for all $t, s > 0$.

   Give and interpretation to your result.

   (b) **Survival Time After a Treatment**. In Problem 5 of Assignment #9 you showed that the survival time, T, after a treatment can be modeled by an exponential random variable with parameter β, where β is the expected time of survival.

   The survival function, $S(t)$, is the probability that a randomly selected person will survive for at least t years after receiving treatment. Compute $S(t)$.

   Suppose that a patient has a 70% probability of surviving at least two years. Estimate the expected survival time of the treatment.