Assignment #8

Due on Wednesday, March 13, 2013

Read Section 4.1 on *Random Variables* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.
Read Chapter on *Diffusion* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.

Background and Definitions The probability density function, $p(x,t)$, for the location, X_t, of a Brownian particle at time t,

$$p(x,t) = \frac{1}{\sqrt{4\pi D t}} \cdot e^{-x^2/(4Dt)}, \quad \text{for } x \in \mathbf{R} \text{ and } t > 0, \tag{1}$$

which was derived in class, is also called the heat kernel. In this set of problems, we derive a few properties of this function.

Do the following problems

1. Verify that p solves the diffusion equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}, \quad \text{for } x \in \mathbf{R} \text{ and } t > 0. \tag{2}$$

2. Explain why $\displaystyle\int_{-\infty}^{\infty} p(x,t)\ dx = 1$, for all $t > 0$. Give an interpretation to this result.

3. Show the following:

   (a) If $x \neq 0$, then $\displaystyle\lim_{t\to 0^+} p(x,t) = 0$.

   (b) If $x = 0$, then $\displaystyle\lim_{t\to 0^+} p(x,t) = +\infty$.

4. Use a mathematical software package to sketch the graph of $x \mapsto p(x,t)$ for several values of $t > 0$.

5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded function that is Riemann integrable on bounded intervals and define

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t) f(y)\ dy, \quad \text{for } x \in \mathbf{R} \text{ and } t > 0.$$

   Explain why u solves the diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad \text{for } x \in \mathbf{R} \text{ and } t > 0.$$