

Assignment #18

Due on Wednesday, April 10, 2013

Read Section 4.1, on *Vector Valued Functions on Euclidean Space*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 4.2, on *Linear Functions*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 2.1 on *Linear Transformations* in Thrall and Tornheim (pp. 32–35).

Background and Definitions

Linear Functions. A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **linear** if

- (i) $T(cv) = cT(v)$ for all scalars c and all $v \in \mathbb{R}^n$, and
- (ii) $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$.

Do the following problems

1. Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows: For each $v \in \mathbb{R}^2$, $T(v)$ is the reflection of the point determined by the coordinates of v , relative to the standard basis in \mathbb{R}^2 , on the line $y = x$ in \mathbb{R}^2 . That is, $T(v)$ determines a point along a line through the point determined by v which is perpendicular to the line $y = x$, and the distance from v to the line $y = x$ is the same as the distance from $T(v)$ to the line $y = x$ (see Figure 1).

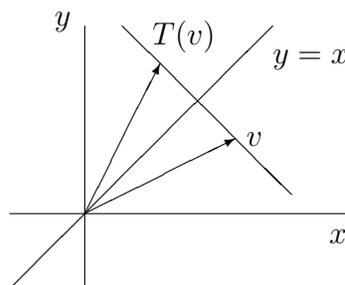


Figure 1: Reflection on the line $y = x$

Prove that T is a linear function.

2. Prove that if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then $T(\mathbf{0}) = \mathbf{0}$, where the first $\mathbf{0}$ denotes the zero-vector in \mathbb{R}^n and the second $\mathbf{0}$ denotes the zero-vector in \mathbb{R}^m .
3. Suppose that $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear and define

$$\mathcal{N}_T = \{v \in \mathbb{R}^n \mid T(v) = \mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero-vector in \mathbb{R}^m .

Prove that \mathcal{N}_T is a subspace of \mathbb{R}^n .

Note: \mathcal{N}_T is called the **null space** of the linear function T .

4. Suppose that $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear and define

$$\mathcal{I}_T = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

Prove that \mathcal{I}_T is a subspace of \mathbb{R}^m .

Note: The set \mathcal{I}_T is called the **image** of the function T . It is also denoted by $T(\mathbb{R}^n)$; thus,

$$T(\mathbb{R}^n) = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

5. Fix $u \in \mathbb{R}^n$ and define $f: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f(v) = \langle u, v \rangle \text{ for all } v \in \mathbb{R}^n.$$

- (a) Prove that f is a linear function.
- (b) Let \mathcal{N}_f denote the null space of f ; that is,

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle u, v \rangle = 0\}.$$

Find the dimension of \mathcal{N}_f for each of the cases: $u = \mathbf{0}$ and $u \neq \mathbf{0}$.