

## Assignment #24

Due on Monday, April 29, 2013

Read Chapter 5, on *The Eigenvalue Problem*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

## Background and Definitions

**Eigenvalues and Eigenvectors.** Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation. A scalar,  $\lambda$ , is said to be an **eigenvalue** of  $T$  if and only if the equation

$$T(v) = \lambda v \tag{1}$$

has a nontrivial solution.

A nontrivial solution,  $v$ , of the equation  $T(v) = \lambda v$  is called an **eigenvector** corresponding to the eigenvalue  $\lambda$ .

Observe that the equation in (1) can also be written as

$$(T - \lambda I)v = \mathbf{0},$$

where  $I: \mathbb{R}^n \rightarrow \mathbb{R}^n$  denotes the identity transformation in  $\mathbb{R}^n$ . Thus,  $\lambda$  is an eigenvalue of  $T$  if and only if the null space of the linear transformation  $T - \lambda I$  is nontrivial; that is  $\mathcal{N}_{T-\lambda I} \neq \{\mathbf{0}\}$ . The null space of  $T - \lambda I$  is called the **eigenspace** of  $T$  corresponding to  $\lambda$  and is denoted by  $E_T(\lambda)$ .

Do the following problems

1. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote a linear transformation in  $\mathbb{R}^2$ . Suppose that  $v_1$  and  $v_2$  are two eigenvectors of  $T$  corresponding to the eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively.

Prove that, if  $\lambda_1 \neq \lambda_2$ , then the set  $\{v_1, v_2\}$  is linearly independent.

Deduce therefore that a linear transformation,  $T$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  cannot have more than two distinct eigenvalues.

2. Show that the rotation  $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  does not have any real eigenvalues unless  $\theta = 0$  or  $\theta = \pi$ .

Give the eigenvalues and corresponding eigenspaces in each case.

3. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where  $A$  is the  $2 \times 2$  matrix

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

Find all the eigenvalue of  $T$  and compute their respective eigenspaces.

4. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where  $A$  is the  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where  $a$  and  $b$  are real constants.

- (a) Show that  $T$  has real eigenvalues.
  - (b) Under what conditions on  $a$  and  $b$  will the eigenvalues obtained in part (a) be distinct eigenvalues?
5. Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation. Prove that  $\lambda = 0$  is an eigenvalue of  $T$  if and only if the matrix representation,  $M_T$ , of  $T$  is singular.