Solutions to Part I of Exam 1

1. Answer the following questions as thoroughly as possible.

   (a) State precisely what it means for the subset, S, of $\mathbb{R}^n$ to be linearly independent.

   Answer: S is linearly independent means that no vector in S is in the span of the other vectors in S. □

   (b) Let W denote a subspace of $\mathbb{R}^n$ and B a subset of W. State precisely what it means for B to be a basis for W.

   Answer: B is a bases for W is B spans W and is linearly independent. □

   (c) Define the dimension of a subspace, W, of $\mathbb{R}^n$.

   Answer: The dimension of W is the number of vectors in any basis of W. □

   (d) Let W denote a subspace of $\mathbb{R}^n$ with ordered basis $B = \{w_1, w_2, \ldots, w_k\}$. For any vector, w in W, define $[w]_B$, the coordinates of w relative to B.

   Answer: The coordinates of $v \in W$ relative to the ordered basis $B = \{w_1, w_2, \ldots, w_k\}$ are the unique set of scalars, $c_1, c_2, \ldots, c_k$ such that

$$w = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

   We write

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

   □

   (e) Given vectors v and w in $\mathbb{R}^n$, state what it means for v and w to be orthogonal.

   Answer: v and w in $\mathbb{R}^n$ are said to be orthogonal if $\langle v, w\rangle = 0$, where $\langle v, w\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. □

2. Let S denote a subset of $\mathbb{R}^n$.

(a) Give a definition of span(S).

Answer: span(S) is the smallest subspace of $\mathbb{R}^n$ that contains S.

Alternatively, span(S) is the collection of all finite linear combinations of vectors in S. □

(b) Let v_1, v_2 and v_3 denote vectors in $\mathbb{R}^n$. Assume that $v_3 \in \text{span}(\{v_1, v_2\})$. Prove that

$$\text{span}(\{v_1, v_2\}) = \text{span}(\{v_1, v_2, v_3\}).$$

Proof: Assume that v_1, v_2 and v_3 are vectors in $\mathbb{R}^3$ with $v_3 \in \text{span}(\{v_1, v_2\})$. From the inclusion $\{v_1, v_2\} \subseteq \{v_1, v_2, v_3\}$ we obtain the inclusion

$$\text{span}(\{v_1, v_2\}) \subseteq \text{span}(\{v_1, v_2, v_3\}), \tag{1}$$

since span($\{v_1, v_2\}$) is the smallest subspace of $\mathbb{R}^n$ that contains $\{v_1, v_2\}$. In order to show the reverse inclusion, observe that the fact that

$$\{v_1, v_2\} \subseteq \text{span}(\{v_1, v_2\})$$

and the assumption that $v_3 \in \text{span}(\{v_1, v_2\})$ imply that

$$\{v_1, v_2, v_3\} \subseteq \text{span}(\{v_1, v_2\}). \tag{2}$$

It follows from (2) that

$$\text{span}(\{v_1, v_2, v_3\}) \subseteq \text{span}(\{v_1, v_2\}), \tag{3}$$

where we have used the fact span($\{v_1, v_2, v_3\}$) is the smallest subspace of $\mathbb{R}^n$ that contains $\{v_1, v_2, v_3\}$.

Combining (1) and (3) yields what we were asked to prove. □

3. Let W denote a subset of $\mathbb{R}^n$.

(a) State precisely what it means for W to be a subspace of $\mathbb{R}^n$.

Answer: W is a subspace of $\mathbb{R}^n$ if it is (i) non–empty, (ii) closed under vector addition, and (iii) closed under scalar multiplication. □

(b) Let $\langle v, w\rangle$ denote the Euclidean inner product in $\mathbb{R}^n$. For a fixed vector u in $\mathbb{R}^n$, define the set

$$W = \{w \in \mathbb{R}^n \mid \langle u, w\rangle = 0\}.$$

Prove that W is a subspace of $\mathbb{R}^n$.

Proof: We show that W is (i) non–empty, (ii) closed under vector addition, and (iii) closed under scalar multiplication.

(i) To see that W is nonempty, observe that $0 \in W$ because $\langle u, \mathbf{0}\rangle = 0$.

(ii) To show that W is closed under vector addition, let w_1 and w_2 be two vectors in W, so that $\langle u, w_1\rangle = 0$ and $\langle u, w_2\rangle = 0$. Then, applying the bi–linearity property of the inner product,

$$\langle u, w_1 + w_2\rangle = \langle u, w_1\rangle + \langle u, w_2\rangle = 0 + 0 = 0;$$

hence, $w_1 + w_2 \in W$.

(iii) To see that W is closed under scalar multiplication, let $w \in W$, so that $\langle u, w\rangle = 0$. Then, for any $t \in \mathbb{R}$,

$$\langle u, tw\rangle = t\langle u, w\rangle = t \cdot 0 = 0,$$

where we have used the bi–linearity property of the inner product. We have therefore shown that $tw \in W$ for all $t \in \mathbb{R}$ and all $w \in W$.

□