Solutions to Part II of Exam 1

1. Let W denote the solution space of the homogenous system

$$\left\{\begin{array}{rcrcrcrcl} x_1 & - & x_2 & & & + & 2x_4 & = & 0 \\ -x_1 & & & + & x_3 & - & 3x_4 & = & 0 \\ x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 0 \\ 2x_1 & - & x_2 & - & x_3 & + & 5x_4 & = & 0. \end{array}\right. \tag{1}$$

(a) Find a basis for W and compute $\dim(W)$.

***Solution**:* Perform elementary row operations on the augmented matrix corresponding to the system in (1),

$$\begin{pmatrix} 1 & -1 & 0 & 2 & | & 0 \\ -1 & 0 & 1 & -3 & | & 0 \\ 1 & -2 & 1 & 1 & | & 0 \\ 2 & -1 & -1 & 5 & | & 0 \end{pmatrix},$$

to get the reduced matrix

$$\begin{pmatrix} 1 & 0 & -1 & 3 & | & 0 \\ 0 & 1 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}. \tag{2}$$

It follows from (2) that the system in (1) is equivalent to the system

$$\left\{\begin{array}{rcrcrcrcl} x_1 & & & - & x_3 & + & 3x_4 & = & 0 \\ & & x_2 & - & x_3 & + & x_4 & = & 0. \end{array}\right. \tag{3}$$

Solving for the leading variables in (3) we obtain

$$\left\{\begin{array}{rcl} x_1 & = & t + 3s \\ x_2 & = & t + s \\ x_3 & = & t \\ x_4 & = & -s, \end{array}\right. \tag{4}$$

where t and s are arbitrary parameters.

It follows from (4) that solutions to (1) are of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \text{for } t, s \in \mathbb{R}.$$

It then follows that

$$W = \text{span}\left(\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\\-1\end{pmatrix}\right\}\right)$$

so that

$$\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\\-1\end{pmatrix}\right\}$$

is a basis for W. Therefore, $\dim(W) = 2$. □

(b) Define $V = \{v \in \mathbb{R}^4 \mid \langle v, w\rangle = 0, \text{ for all } w \in W\}$, where $\langle v, w\rangle$ denotes the Euclidean inner product of v and w in $\mathbb{R}^4$.

Show that V is subspace of $\mathbb{R}^4$, give a basis for V, and compute $\dim(V)$.

***Solution**:* We show that V is (i) non–empty, (ii) closed under vector addition, and (iii) closed under scalar multiplication.

(i) To see that V is nonempty, observe that $0 \in V$ because $\langle \mathbf{0}, w\rangle = 0$ for all $w \in W$.

(ii) To show that V is closed under vector addition, let v_1 and v_2 be two vectors in V, so that $\langle v_1, w\rangle = 0$ and $\langle v_2, w\rangle = 0$, for all $w \in W$. Then, applying the bi–linearity property of the inner product,

$$\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle = 0 + 0 = 0,$$

for all $w \in W$; so that $v_1 + v_2 \in V$.

(iii) To see that V is closed under scalar multiplication, let $v \in V$, so that $\langle v, w\rangle = 0$ for all $w \in W$. Then, for any $t \in \mathbb{R}$,

$$\langle tv, w\rangle = t\langle v, w\rangle = t \cdot 0 = 0,$$

for all $w \in W$, where we have used the bi–linearity property of the inner product. We have therefore shown that $tv \in V$ for all $t \in \mathbb{R}$ and all $v \in V$.

Thus, V is a subspace of $\mathbb{R}^4$.

Next, we determine subspace V.

Let

$$w_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix}3\\1\\0\\-1\end{pmatrix},$$

so that, according to the result in part (b), $\{w_1, w_2\}$ is a basis for W. Thus, the vectors in V must be orthogonal to both w_1 and w_2. Hence, if

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

is an arbitrary vector in V, it must be the case that

$$\langle w_1, v\rangle = 0 \quad \text{and} \quad \langle w_2, v\rangle = 0,$$

or

$$\left\{ \begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & & & = & 0 \\ 3x_1 & + & x_2 & - & & & x_4 & = & 0. \end{array} \right. \tag{5}$$

We solve the system in (5) by performing elementary row operations to the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \end{array} \right)$$

leading to

$$\left(\begin{array}{cccc|c} 1 & 0 & -1/2 & -1/2 & 0 \\ 0 & 1 & -3/2 & 1/2 & 0 \end{array} \right);$$

so that the system in (5) is equivalent to the system

$$\left\{ \begin{array}{rcrcl} x_1 & - & \frac{1}{2}x_3 - \frac{1}{2}x_4 & = & 0 \\ & x_2 & + \frac{3}{2}x_3 + \frac{1}{2}x_4 & = & 0. \end{array} \right. \tag{6}$$

Solving for the leading variables in (6) we obtain

$$\left\{ \begin{array}{rcl} x_1 & = & t + s \\ x_2 & = & -3t - s \\ x_3 & = & 2t \\ x_4 & = & 2s, \end{array} \right.$$

where t and s are arbitrary parameters; so that V consists of vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \quad \text{for } t, s \in \mathbb{R}.$$

Thus,

$$V = \text{span}\left(\left\{\begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}\right\}\right),$$

so that

$$\left\{\begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}\right\}$$

is a basis for V and $\dim(V) = 2$. □

2. Let $A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ -1 & 0 & -2 & -1 \\ 0 & 1 & 1 & -1 \\ 2 & -3 & 1 & 5 \end{pmatrix}$, and denote by C_A the span of the columns of the matrix A.

   (a) Give a basis for C_A and compute $\dim(C_A)$.

   ***Solution**:* Denote the columns of A by v_1, v_2, v_3 and v_4, respectively, and assume that

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \mathbf{0}. \tag{7}$$

The vector equation in (7) is equivalent to the homogeneous system

$$\left\{\begin{array}{rcrcrcrcl} c_1 & - & c_2 & + & c_3 & + & 2c_4 & = & 0 \\ -c_1 & & & - & 2c_3 & - & c_4 & = & 0 \\ & & c_2 & + & c_3 & - & c_4 & = & 0 \\ 2c_1 & - & 3c_2 & + & c_3 & + & 5c_4 & = & 0. \end{array}\right. \tag{8}$$

We solve the system in (8) by reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 & 2 & | & 0 \\ -1 & 0 & -2 & -1 & | & 0 \\ 0 & 1 & 1 & -1 & | & 0 \\ 2 & -3 & 1 & 5 & | & 0 \end{pmatrix}$$

to the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix},$$

so that the system in (8) is equivalent to the system

$$\left\{\begin{array}{rcrcrcl} c_1 & + & 2c_3 & + & c_4 & = & 0 \\ c_2 & + & c_3 & - & c_4 & = & 0. \end{array}\right. \tag{9}$$

Solving the system in (9) for the leading variables yields

$$\left\{\begin{array}{rcl} c_1 & = & 2t + s \\ c_2 & = & t - s \\ c_3 & = & -t \\ c_4 & = & -s, \end{array}\right. \tag{10}$$

for arbitrary parameters t and s.

It follows from (10) that the vector equation in (7) has infinitely many solutions and therefore the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

Next, we find a subset of $\{v_1, v_2, v_3, v_4\}$ that is linearly independent and also spans C_A.

Taking $t = 1$ and $s = 0$ in (10) yields the following relation from the vector equation in (7)

$$2v_1 + v_2 - v_3 = \mathbf{0}. \tag{11}$$

Similarly, taking $t = 0$ and $s = 1$ in (10) yields the relation

$$v_1 - v_2 - v_4 = \mathbf{0}. \tag{12}$$

□

Now, it follows from (11) that

$$v_3 = 2v_1 + v_2,$$

so that

$$v_3 \in \text{span}(\{v_1, v_2\}). \tag{13}$$

Similarly, using (12), we obtain that

$$v_4 \in \text{span}(\{v_1, v_2\}). \tag{14}$$

It follows from (13) and (14) that

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}(\{v_1, v_2\}),$$

so that

$$C_A \subseteq \text{span}(\{v_1, v_2\}), \tag{15}$$

since C_A is the smallest subspace of $\mathbb{R}^4$ that contains the columns of A. On the other hand, since v_1 and v_2 are columns of A, we also get that

$$\text{span}(\{v_1, v_2\}) \subseteq C_A, \tag{16}$$

Putting (15) and (16) together, we get that

$$C_A = \text{span}(\{v_1, v_2\}),$$

where the set $\{v_1, v_2\}$ is linearly independent. Hence, the first two columns of A form a basis for C_A and, therefore, $\dim(A) = 2$.

(b) Determine whether or not the vector $v = \begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix}$ is in C_A.

***Solution**:* Since $C_A = \text{span}(\{v_1, v_2\})$, by part (a), we seek scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix}. \tag{17}$$

We attempt to solve the equation in (17) by performing elementary row operations on the augmented matrix

$$\begin{pmatrix} 1 & -1 & | & 4 \\ -1 & 0 & | & 7 \\ 0 & 1 & | & 7 \\ 2 & -3 & | & 4 \end{pmatrix}.$$

We are led to

$$\begin{pmatrix} 1 & -1 & | & 4 \\ 0 & 1 & | & -11 \\ 0 & 0 & | & 18 \\ 0 & 0 & | & -15 \end{pmatrix}. \tag{18}$$

Note that the last two rows of the augmented matrix in (18) lead to false statements. Hence, the equation in (17) has no solutions. Thus, the vector $\begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix}$ is not in the span of the columns of A. □

(c) Given an arbitrary vector $v = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ in $\mathbb{R}^4$, determine conditions on x, y, z and w that will guarantee that $v \in C_A$.

Solution: We seek scalars c_1 and c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \tag{19}$$

We attempt to solve the equation in (19) by performing elementary row operations on the augmented matrix

$$\left(\begin{array}{rr|c} 1 & -1 & x \\ -1 & 0 & y \\ 0 & 1 & z \\ 2 & -3 & w \end{array}\right).$$

We are led to

$$\left(\begin{array}{rr|c} 1 & -1 & x \\ 0 & 1 & -x-y \\ 0 & 0 & x+y+z \\ 0 & 0 & -3x-y+w \end{array}\right). \tag{20}$$

In order for the equation in (19) to yield a consistent system, all the entries in the last two rows of the augmented matrix in (20) must be 0. Hence, in order for the equation (19) to have solutions, it must be the case that

$$\left\{ \begin{array}{rcrcrcrcl} x & + & y & + & z & & & = & 0 \\ 3x & + & y & & & - & w & = & 0. \end{array} \right.$$

$\square$