Review Problems for Exam 1

1. Consider the set $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

   (a) Show that B is a basis for $\mathbb{R}^2$.

   (b) Give the coordinates of the vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ relative to B. Interpret your result geometrically.

2. Give a basis for the span of the following set of vectors in $\mathbb{R}^4$

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

3. Find a basis for the solution space of the system

$$\left\{ \begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & x_2 & & & - & 2x_4 & = & 0 \\ -x_1 & & & + & x_3 & + & x_4 & = & 0, \end{array} \right.$$

   and compute its dimension.

4. Prove that any set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

5. Let v_1 and v_2 denote vectors in $\mathbb{R}^n$.

   (a) Show that if the set $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^n$, then so is the set $\{v_1, cv_1 + v_2\}$, where c is a scalar, and, conversely, if $\{v_1, cv_1 + v_2\}$ is linearly independent, then so is $\{v_1, v_2\}$.

   (b) Show that $\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}$.

6. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent subset of $\mathbb{R}^n$. Suppose there exists $v \in \mathbb{R}^n$ such that $v \notin \text{span}(S)$. Show that the set $S \cup \{v\}$ is linearly independent.

7. Let S denote a nonempty subset of $\mathbb{R}^n$. Assume that there exists $v \in S$ such that $v \in \text{span}(S \backslash \{v\})$. Show that

$$\text{span}(S \backslash \{v\}) = \text{span}(S).$$

8. Let J and H be planes in $\mathbb{R}^3$ given by

$$J = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 2x + 3y - 6z = 0 \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 2y + z = 0 \right\}.$$

   (a) Give bases for J and H and compute their dimensions.

   (b) Give a basis for the subspace $J \cap H$ and compute $\dim(J \cap H)$.

9. Let W be a subspace of $\mathbb{R}^n$.

   (a) Prove that if $v \in W$ and $v \neq \mathbf{0}$, then $rv = sv$ implies that $r = s$, where r and s are scalars.

   (b) Prove that if W has more than one element, then W has infinitely many elements.

10. Let W be a subspace of $\mathbb{R}^n$ and S_1 and S_2 be subsets of W.

   (a) Show that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

   (b) Give an example in which $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$.

11. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. We write $W_1 \oplus W_2$ for the subspace $W_1 + W_2$ for the special case in which $W_1 \cap W_2 = \{\mathbf{0}\}$. Show that every vector $v \in W_1 \oplus W_2$ can be written in the form $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$, in one and only one way; that is, if $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$, then $u_1 = v_1$ and $u_2 = v_2$.

12. Let $v_1, v_2, \ldots, v_k$ be nonzero vectors in $\mathbb{R}^n$ that are mutually orthogonal; that is $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Prove that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

13. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x - y + 2z = 0 \right\}$. Find a basis for W consisting of vectors that are mutually orthogonal.

14. Let $v \in \mathbb{R}^n$ and define $W = \{w \in \mathbb{R}^n \mid \langle w, v \rangle = 0\}$.

   (a) Prove that W is a subspace of $\mathbb{R}^n$.

   (b) Suppose that $v \neq \mathbf{0}$ and compute $\dim(W)$.