Solutions to Review Problems for Exam 1

1. Consider the set $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

   (a) Show that B is a basis for $\mathbb{R}^2$.

   Proof: Given that $\dim(\mathbb{R}^2) = 2$ and that B contains two vectors, to prove that B is a basis for $\mathbb{R}^2$, it suffices to prove that B is linearly independent. Thus, consider the vector equation

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

   which is equivalent to the system

$$\begin{cases} c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0. \end{cases} \tag{2}$$

   The system in (2) can be solved to yield the unique solution $c_1 = c_2 = 0$. Hence, the vector equation in (1) has only the trivial solution, and therefore B is linearly independent. □

   (b) Give the coordinates of the vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ relative to B. Interpret your result geometrically.

   Solution: We look for scalars, c_1 and c_2, such that

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

   This is equivalent to solving the system

$$\begin{cases} c_1 - c_2 &= 1 \\ c_1 + c_2 &= 0. \end{cases}$$

   To solve this system, we may reduce the corresponding augmented matrix,

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 0 \end{array} \right),$$

   to

$$\left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array} \right).$$

We therefore get that the coordinate vector of $\begin{pmatrix}1\\0\end{pmatrix}$ relative to B is

$$\left[\begin{pmatrix}1\\0\end{pmatrix}\right]_B = \begin{pmatrix}1/2\\-1/2\end{pmatrix}.$$

Denote the vectors in B by v_1 and v_2, respectively and in that order, and denote the vector $\begin{pmatrix}1\\0\end{pmatrix}$ by v. Figure 1 shows the vector v as the sum of the vectors $\frac{1}{2}v_1$ and $-\frac{1}{2}v_2$.

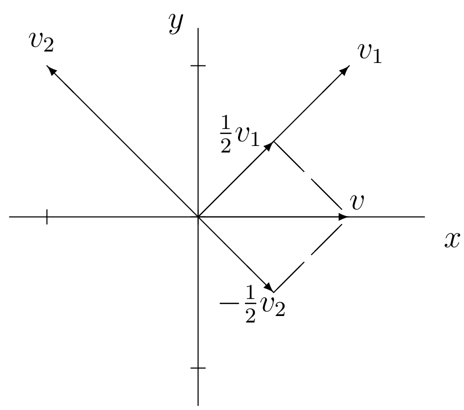

Figure 1: Coordinates relative to B

□

2. Give a basis for the span of the following set of vectors in $\mathbb{R}^4$

$$\left\{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-2\\0\\3\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\6\\-3\end{pmatrix}, \begin{pmatrix}1\\1\\-4\\1\end{pmatrix}\right\}.$$

***Solution**:* Denote the vectors in the set

$$\left\{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-2\\0\\3\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\6\\-3\end{pmatrix}, \begin{pmatrix}1\\1\\-4\\1\end{pmatrix}\right\}$$

by v_1, v_2, v_3 and v_4, respectively, we look for a linear vector relation of the form

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \mathbf{0}. \tag{4}$$

This leads to the system

$$\left\{\begin{array}{rcl} c_1 - 2c_2 + c_3 + c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0 \\ c_1 + 3c_2 + 6c_3 - 4c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0. \end{array}\right. \tag{5}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 0 \\ -1 & 0 & -3 & 1 & 0 \\ 1 & 3 & 6 & -4 & 0 \\ -1 & 0 & -3 & 1 & 0 \end{array}\right).$$

We can reduce this matrix to

$$\left(\begin{array}{rrrr|r} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which is in reduced row–echelon form. We therefore get that the system in (5) is equivalent to the system

$$\left\{\begin{array}{rcl} c_1 + 3c_3 - c_4 & = & 0 \\ c_2 + c_3 - c_4 & = & 0. \end{array}\right. \tag{6}$$

Solving for the leading variables in (6) yields the solutions

$$\left\{\begin{array}{rcl} c_1 & = & 3t + s \\ c_2 & = & t + s \\ c_3 & = & -t \\ c_4 & = & s, \end{array}\right. \tag{7}$$

where t and s are arbitrary parameters. Taking $t = 1$ and $s = 0$ in (7) yields from (4) the linear relation

$$3v_1 + v_2 - v_3 = \mathbf{0},$$

which shows that $v_3 = -3v_1 - v_2$; that is, $v_3 \in \text{span}\{v_1, v_2\}$.

Similarly, taking $t = 0$ and $s = 1$ in (7) yields

$$v_1 + v_2 + v_4 = \mathbf{0},$$

which shows that $v_4 = -v_1 - v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.
We then have that both v_3 and v_4 are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is, $\{v_1, v_2\}$ spans $\text{span}\{v_1, v_2, v_3, v_4\}$.
To see that $\{v_1, v_2\}$ is linearly independent, observe that v_1 and v_2 are not multiples of each other. We therefore conclude that $\{v_1, v_2\}$ is a basis for $\text{span}\{v_1, v_2, v_3, v_4\}$. $\square$

3. Find a basis for the solution space of the system

$$\begin{cases} x_1 - x_2 + x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_4 = 0 \\ -x_1 + x_3 + x_4 = 0, \end{cases} \tag{8}$$

and compute its dimension.

***Solution**:* We first find the solution space, W, of the system. In order to do this, we reduce the augmented matrix of this system,

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \quad \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 2 & -1 & 0 & -2 & 0 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right),$$

to its reduced row–echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Consequently, the system in (8) is equivalent to the system

$$\left\{\begin{array}{rcrcrcl} x_1 & - & x_3 & - x_4 & = & 0 \\ & x_2 - & 2x_3 & & = & 0. \end{array}\right. \tag{9}$$

Solving for the leading variables in the system in (9) we obtain the solutions

$$\left\{\begin{array}{rcl} x_1 & = & t + s \\ x_2 & = & 2t \\ x_3 & = & t \\ x_4 & = & s, \end{array}\right.$$

where t and s are arbitrary parameters. I then follows that the solution space of system (9) is

$$W = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Hence

$$\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

is a basis for W and therefore $\dim(W) = 2$. □

4. Prove that any set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

Proof: Let v_1 , v_2, v_3 and v_4 denote four vectors in $\mathbb{R}^3$ and write

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \quad \text{and} \quad v_4 = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}.$$

Consider the vector equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4, = \mathbf{0}. \tag{10}$$

This equation translates into the homogeneous system

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + a_{14}c_4 &= 0 \\ a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + a_{24}c_4 &= 0 \\ a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + a_{34}c_4 &= 0, \end{cases} \tag{11}$$

of 3 linear equations in 4 unknowns. It then follows from the Fundamental Theorem for Homogeneous Linear Systems that system (11) has infinitely many solutions. Consequently, the vector equation in (10) has a nontrivial solution, and therefore the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. □

5. Show that if the set $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^n$, then so is the set $\{v_1, cv_1 + v_2\}$, where c is a scalar, and, conversely, if $\{v_1, cv_1 + v_2\}$ is linearly independent, then so is $\{v_1, v_2\}$. Show also that $\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}$.

   (a) First we prove that $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^n$, then so is the set $\{v_1, cv_1 + v_2\}$,

   Proof: Assume that $\{v_1, v_2\}$ is a linearly independent and consider the vector equation

$$c_1 v_1 + c_2(cv_1 + v_2) = \mathbf{0}. \tag{12}$$

   Applying the distributive and associative properties, the equation in (12) turns into

$$(c_1 + cc_2)v_1 + c_2 v_2 = \mathbf{0}. \tag{13}$$

   It follows from (13) and the linear independence of $\{v_1, v_2\}$ that

$$\begin{cases} c_1 + cc_2 &= 0 \\ c_2 &= 0. \end{cases} \tag{14}$$

   The system in (14) has only the trivial solution: $c_2 = c_1 = 0$. Hence, the vector equation in (12) has only the trivial solution and therefore the set $\{v_1, cv_1 + v_2\}$ is linearly independent. □

   Next, we prove the converse: If $\{v_1, cv_1 + v_2\}$ is linearly independent, then $\{v_1, v_2\}$ is a linearly independent.

   Proof: Assume that $\{v_1, cv_1 + v_2\}$ is a linearly independent and consider the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}. \tag{15}$$

Adding $\mathbf{0} = cc_2v_1 - cc_2v_1$ to the left–hand side of the equation in (15) and applying the distributive and associative properties we get

$$(c_1 - cc_2)v_1 + c_2(cv_1 + v_2) = \mathbf{0}. \tag{16}$$

It follows from (16) and the linear independence of $\{v_1, cv_1 + v_2\}$ that

$$\left\{ \begin{array}{rcl} c_1 - cc_2 & = & 0 \\ c_2 & = & 0. \end{array} \right. \tag{17}$$

The system in (17) has only the trivial solution: $c_2 = c_1 = 0$. Hence, the vector equation in (15) has only the trivial solution and therefore the set $\{v_1, v_2\}$ is linearly independent. □

(b) We prove that that $\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}$.

Proof: Let $W = \text{span}\{v_1, v_2\}$. Then, W is a subspace which contains v_1 and v_2 and all their linear combinations; in particular, $cv_1 + v_2 \in W$. We then have that

$$\{v_1, cv_1 + v_2\} \subseteq W.$$

It then follows that

$$\text{span}\{v_1, cv_1 + v_2\} \subseteq W, \tag{18}$$

since $\text{span}\{v_1, cv_1 + v_2\}$ is the smallest subspace of $\mathbb{R}^n$ which contains $\{v_1, cv_1 + v_2\}$. Ont the other hand, for any $u \in W$ there exist scalars c_1 and c_2 such that

$$u = c_1v_1 + c_2v_2.$$

Consequently,

$$\begin{array}{rcl} u & = & c_1v_1 + c_2v_2 + cc_2v_1 - cc_2v_1 \\ & = & (c_1 - cc_2)v_1 + c_2(cv_1 + v_2), \end{array}$$

which shows that $u \in \text{span}\{v_1, cv_1 + v_2\}$; thus,

$$u \in W \Rightarrow u \in \text{span}\{v_1, cv_1 + v_2\},$$

or

$$W \subseteq \text{span}\{v_1, cv_1 + v_2\}.$$

Combining this with (18) yields that

$$W = \text{span}\{v_1, cv_1 + v_2\}.$$

□

6. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent subset of $\mathbb{R}^n$. Suppose there exists $v \in \mathbb{R}^n$ such that $v \notin \text{span}(S)$. Show that the set $S \cup \{v\}$ is linearly independent.

   Proof: Assume that $S = \{v_1, v_2, \ldots, v_k\}$ is a linearly independent subset of $\mathbb{R}^n$ and that $v \in \mathbb{R}^n$ is such that $v \notin \text{span}(S)$. Suppose that

   $$c_1v_1 + c_2v_2 + \cdots + c_kv_k + cv = 0. \tag{19}$$

   We first see that c in (19) must be 0; otherwise, $c \neq 0$ and we can solve for v in (19) to get that

   $$v = -\frac{c_1}{c}v_1 - \frac{c_2}{c}v_2 - \cdots - \frac{c_k}{c}v_k,$$

   which shows that $v \in\in \text{span}(S)$; this contradicts the assumption that $v \notin \text{span}(S)$. Hence, $c = 0$ and so we obtain from (19) that

   $$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0. \tag{20}$$

   It follows from (20) and the assumption that S is linearly independent that

   $$c_1 = c_2 = \ldots = c_k = 0.$$

   We have therefore shown that (19) implies that

   $$c_1 = c_2 = \ldots = c_k = c = 0.$$

   Hence, the set $S \cup \{v\}$ is linearly independent. □

7. Let S denote a nonempty subset of $\mathbb{R}^n$. Assume that there exists $v \in S$ such that $v \in \text{span}(S\backslash\{v\})$. Show that

   $$\text{span}(S\backslash\{v\}) = \text{span}(S).$$

   Proof: Let $S \subseteq \mathbb{R}^n$ and assume that there exists $v \in S$ such that $v \in \text{span}(S\backslash\{v\})$. First observe that $S\backslash\{v\} \subseteq S$, so that

   $$S\backslash\{v\} \subseteq \text{span}(S).$$

   Thus,

   $$\text{span}(S\backslash\{v\}) \subseteq \text{span}(S) \tag{21}$$

   because $\text{span}(S\backslash\{v\})$ is the smallest subspace of $\mathbb{R}^n$ that contains $S\backslash\{v\}$.

Next, let $w \in S$. We have two possibilities: (i) $w \neq v$, or (ii) $w = v$. If $w \neq v$, then $w \in \text{span}(S\backslash\{v\})$; on the other hand, if $w = v$, then, by assumption, $w \in \text{span}(S\backslash\{v\})$. In both cases, $w \in \text{span}(S\backslash\{v\})$. Thus,

$$S \subseteq \text{span}(S\backslash\{v\}).$$

It then follows that

$$\text{span}(S) \subseteq \text{span}(S\backslash\{v\}) \tag{22}$$

because $\text{span}(S)$ is the smallest subspace of $\mathbb{R}^n$ that contains S.

Combining (21) and (22) yields what we were asked to prove. □

8. Let J and H be planes in $\mathbb{R}^3$ given by

$$J = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 2x + 3y - 6z = 0 \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 2y + z = 0 \right\}.$$

(a) Give bases for J and H and compute their dimensions.

***Solution**:* To find a basis for J, we solve the equation

$$2x + 3y + z = 0$$

to get the solution space $J = \text{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$. Thus, the set

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$$

is a basis for J and so $\dim(J) = 2$.
Similarly, for H, we solve

$$x - 2y + z = 0$$

and obtain that

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is a basis for H; thus, $\dim(H) = 2$. □

(b) Give a basis for the subspace $J \cap H$ and compute $\dim(J \cap H)$.

Solution: Vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the intersection of J and H if they are solutions to the system of equations

$$\left\{ \begin{array}{rcl} 2x + 3y - 6z & = & 0 \\ x - 2y + z & = & 0. \end{array} \right. \tag{23}$$

Thus, to find $J \cap H$, we may elementary row operations on the augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \end{array} \quad \left(\begin{array}{rrr|r} 2 & 3 & -6 & 0 \\ 1 & -2 & 1 & 0 \end{array} \right)$$

to obtain the reduced matrix

$$\left(\begin{array}{rrr|r} 1 & 0 & -9/7 & 0 \\ 0 & 1 & -8/7 & 0 \end{array} \right).$$

Thus, the system in (23) is equivalent to

$$\left\{ \begin{array}{rcl} x - \frac{9}{7}z & = & 0 \\ y - \frac{8}{7}z & = & 0, \end{array} \right. \tag{24}$$

Solving for the leading variables in system (24) and setting $z = 7t$, where t is an arbitrary parameter, wee obtain that

$$J \cap H = \operatorname{span} \left\{ \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \right\}.$$

Thus, the set

$$\left\{ \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \right\}$$

is a basis for $J \cap H$ and, therefore, $\dim(J \cap H) = 1$. □

9. Let W be a subspace of $\mathbb{R}^n$.

(a) Prove that if $v \in W$ and $v \neq \mathbf{0}$, then $rv = sv$ implies that $r = s$, where r and s are scalars.

Proof: Suppose that $v \in W$, where W is a subspace of $\mathbb{R}^n$, and that $v \neq \mathbf{0}$. Suppose also that

$$rv = sv \tag{25}$$

for some scalars r and s. Add $-sv$ on both sides of the vector equation in (25) and apply the distributive property to obtain

$$(r - s)v = \mathbf{0}. \tag{26}$$

Taking the Euclidean inner product with v of both sides of (26) yields

$$(r - s)\langle v, v\rangle = 0, \tag{27}$$

where we have used the bi–linearity of the inner product. It then follows form (27), the positive definiteness of the inner product, and the assumption that $v \neq \mathbf{0}$, that

$$r - s = 0$$

and therefore $r = s$, which was to be shown. □

(b) Prove that if W has more than one element, then W has infinitely many elements.

Proof: Since W has at least two elements, there has to be a vector, v, in W such that $v \neq \mathbf{0}$. Now, for any $t \in \mathbb{R}$, $tv \in W$ because W is closed under scalar multiplication. By part (a), $t_1 v \neq t_2 v$ for any $t_1 \neq t_2$. Consequently, W contains infinitely many vectors. □

10. Let W be a subspace of $\mathbb{R}^n$ and S_1 and S_2 be subsets of W.

(a) Show that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

Proof: First observe that $S_1 \cap S_2 \subseteq S_1$ and $S_1 \cap S_2 \subseteq S_2$. Consequently,

$$\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \quad \text{and} \quad \text{span}(S_1 \cap S_2) \subseteq \text{span}(S_2).$$

It then follows that

$$\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2),$$

which was to be shown. □

(b) Give an example in which $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$.

Solution: Let $S_1 = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $S_2 = \left\{\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\}$. Then, $S_1 \cap S_2 = \emptyset$ so that $\text{span}(S_1 \cap S_2) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^2$.

On the other hand,

$$\text{span}(S_1) = \text{span}(S_2)$$

because $\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence,

$$\text{span}(S_1) \cap \text{span}(S_2) = \left\{\begin{pmatrix} t \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t \in \mathbb{R}\right\} \neq \{\mathbf{0}\}.$$

□

11. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. We write $W_1 \oplus W_2$ for the subspace $W_1 + W_2$ for the special case in which $V = W_1 \cap W_2 = \{\mathbf{0}\}$. Show that every vector $v \in W_1 \oplus W_2$ can be written in the form $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$, in one and only one way; that is, if $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$, then $u_1 = v_1$ and $u_2 = v_2$.

    Proof: Suppose that W_1 and W_2 are two subspaces of $\mathbb{R}^n$ which have only the zero vector in common; that is, $W_1 \cap W_2 = \{\mathbf{0}\}$. Let v be any vector in $W_1 + W_2$. Then, $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$. Suppose that v can also be written as $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$. Then,

$$v_1 + v_2 = u_1 + u_2,$$

from which we get that

$$v_1 - u_1 = v_2 - u_2, \tag{28}$$

where $v_1 - u_1 \in W_1$ and $v_2 - u_2 \in W_2$ since W_1 and W_2 are subspaces of $\mathbb{R}^n$. It also follows from (28) that $v_1 - u_1 \in W_2$. Thus, $v_1 - u_1 \in W_1 \cap W_2 = \{\mathbf{0}\}$, which implies that

$$v_1 - u_1 = \mathbf{0},$$

or

$$v_1 = u_1.$$

Similarly, we get that $v_2 = u_2$. □

12. Let $v_1, v_2, \ldots, v_k$ be nonzero vectors in $\mathbb{R}^n$ that are mutually orthogonal; that is $\langle v_i, v_j \rangle = 0$ for $i \neq j$. Prove that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

    Proof: Assume that $v_1, v_2, \ldots, v_k$ are nonzero vectors in $\mathbb{R}^n$ that are mutually orthogonal.

    Suppose that
$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0. \tag{29}$$
    Take inner product with v_1 on both sides of (29) to get
$$\langle c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, v_1 \rangle = \langle 0, v_1 \rangle. \tag{30}$$
    Next, apply the bi–linearity of the inner product on the left–hand side of (30) to get
$$c_1\langle v_1, v_1 \rangle + c_2\langle v_2, v_1 \rangle + \cdots + c_k\langle v_k, v_1 \rangle = 0,$$
    so that
$$c_1\|v_1\|^2 = 0, \tag{31}$$
    where we have used the orthogonality assumption.

    It follows from (31) and the assumption that $v_1 \neq 0$ that $c_1 = 0$. Similarly, taking the inner product with v_j, for $j = 2, 3, \ldots, k$, on both sides of (29) yields that $c_j = 0$ for $j = 2, 3, \ldots, k$. We have therefore shown that (29) implies that
$$c_1 = c_2 = \cdots = c_k = 0.$$
    Hence, the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. □

13. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x - y + 2z = 0 \right\}$. Find a basis for W consisting of vectors that are mutually orthogonal.

    ***Solution**:* We first note that $W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

    Set
$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$
    We then have that $\{v_1, v_2\}$ is a basis for W and $\dim(W) = 2$.

Next, we look for a basis, $\{w_1, w_2\}$, of W made up of orthogonal vectors.
Set $w_1 = v_1$ and look for $w \in \text{span}(\{v_1, v_2\})$ with the property that

$$\langle w, v_1\rangle = 0. \tag{32}$$

Write $w = c_1v_1 + c_2v_2$ and substitute into (32) to get

$$\langle c_1v_1 + c_2v_2, v_1\rangle = 0,$$

or

$$c_1\langle v_1, v_1\rangle + c_2\langle v_2, v_1\rangle = 0, \tag{33}$$

where we have used the bi–linearity of the inner product.
Next, compute

$$\langle v_1, v_1\rangle = 2 \quad \text{ and } \quad \langle v_2, v_1\rangle = -2$$

and substitute into (33) to get the equation

$$2c_1 - 2c_2 = 0,$$

or

$$c_1 - c_2 = 0. \tag{34}$$

The equation in (34) has infinitely many solutions given by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{35}$$

Taking $t = 1$ in (35) we get that $c_1 = c_2 = 1$, so that

$$w = v_1 + v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

is lies in W and is orthogonal to w_1. Set

$$w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Then, $\{w_1, w_2\}$ is a basis for W made up of orthogonal vectors. □

14. Let $v \in \mathbb{R}^n$ and define $W = \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\}$.

    (a) Prove that W is a subspace of $\mathbb{R}^n$.

    Proof: First, observe that $W \neq \emptyset$ because $\langle \mathbf{0}, v\rangle = 0$ and therefore $\mathbf{0} \in W$; thus, W is nonempty.

    Next, we show that W is closed under addition and scalar multiplication.

    To see that W is closed under scalar multiplication, observe that, by the bi–linearity property of the inner product, if $w \in W$, then

    $$\langle\langle v, tw\rangle = t\langle v, w\rangle = t \cdot 0 = 0$$

    for all $t \in \mathbb{R}$.

    To show that W is closed under vector addition, let w_1 and w_2 be two vectors in W. Then, applying the bi–linearity property of the inner product again,

    $$\langle w_1 + w_2, v\rangle = \langle w_1, v\rangle + \langle w_2, v\rangle = 0 + 0 = 0;$$

    hence, $w_1 + w_2 \in W$. □

    (b) Suppose that $v \neq \mathbf{0}$ and compute $\dim(W)$.

    ***Solution**:* Let $B = \{w_1, w_2, \ldots w_k\}$ be a basis for W. Then, $\dim(W) = k$ and we would like to determine what k is.

    First note that $v \notin \text{span}(B)$. For, suppose that $v \in \text{span}(B) = W$, then

    $$\langle v, v\rangle = 0.$$

    Thus, by the positive definiteness of the Euclidean inner product, it follows that $v = \mathbf{0}$, but we are assuming that $v \neq \mathbf{0}$. Consequently, the set

    $$B \cup \{v\} = \{w_1, w_2, \ldots w_k, v\}$$

    is linearly independent. We claim that $B \cup \{v\}$ also spans $\mathbb{R}^n$. To see why this is so, let $u \in \mathbb{R}^n$ be any vector in $\mathbb{R}^n$, and let

    $$t = \frac{\langle u, v\rangle}{\|v\|^2}.$$

    Write

    $$u = tv + (u - tv),$$

and observe that $u - tv \in W$. To see why this is so, compute

$$\begin{aligned} \langle u - tv, v\rangle &= \langle u, v\rangle - t\langle v, v\rangle \\ &= \langle u, v\rangle - t\|v\|^2 \\ &= \langle u, v\rangle - \frac{\langle u, v\rangle}{\|v\|^2}\|v\|^2 \\ &= \langle u, v\rangle - \ \langle u, v\rangle \\ &= 0. \end{aligned}$$

Thus, $u - tv \in W$. It then follows that there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$u - tv = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

Thus,

$$u = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k + tv,$$

which shows that $u \in \text{span}(B \cup \{v\})$. Consequently, $B \cup \{v\}$ spans $\mathbb{R}^n$. Therefore, since $B \cup \{v\}$ is also linearly independent, it forms a basis for $\mathbb{R}^n$. We then have that $B \cup \{v\}$ must have n vectors in it, since $\dim(\mathbb{R}^n) = n$; that is,

$$k + 1 = n,$$

from which we get that

$$\dim(W) = n - 1.$$

□