Solutions to Part I of Exam 2

1. **Answers:**

   (a) The function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear if
   $$f(v+w) = f(v) + f(w), \quad \text{for all } v, w \in \mathbb{R}^n,$$
   and
   $$f(cv) = cf(v), \quad \text{for all } v \in \mathbb{R}^n \text{ and all scalars } c.$$

   (b) An $n \times n$ matrix, A, is invertible if there exists an $n \times n$ matrix, B, such that
   $$BA = AB = I,$$
   where I denotes the $n \times n$ identity matrix.

   (c) An $m \times n$ matrix, A, is singular if the equation
   $$Ax = \mathbf{0}$$
   has nontrivial solutions.

   (d) A scalar, λ, is an eigenvalue of the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ if the equation
   $$T(v) = \lambda v$$
   has nontrivial solutions.

   (e) If λ is an eigenvalue of a linear transformation, $T\colon \mathbb{R}^n \to \mathbb{R}^n$, then the eigenspace of T corresponding to λ, $E_T(\lambda)$, is the set of solutions to the equation
   $$T(v) = \lambda v.$$

2. Let Q denote an $n \times n$ matrix.

   (a) State what it means for Q to be an orthogonal matrix.

   Answer: The $n \times n$ matrix Q is orthogonal means that $Q^TQ = I$. □

   (b) Show that if Q is orthogonal, then $|\det(Q)| = 1$.

   Solution: Assume that Q is orthogonal. Then,
   $$Q^TQ = I. \tag{1}$$
   Taking the determinant on both sides of (1) yields
   $$\det(Q^TQ) = \det(I),$$

from which we get

$$\det(Q^T)\det(Q) = 1,$$

or

$$\det(Q)\det(Q) = 1,$$

since $\det(Q^T) = \det(Q)$. Thus,

$$\det(Q)^2 = 1. \tag{2}$$

Taking the positive square root on both sides of (2) yields

$$|\det(Q)| = 1,$$

which was to be shown. □

(c) Show that if Q is orthogonal, then Q is invertible and give a formula for computing Q^{-1}.

***Solution**:* Assume that Q is orthogonal. Then,

$$Q^TQ = I,$$

which shows that Q has a left–inverse Q^T. It then follows that Q is invertible with

$$Q^{-1} = Q^T.$$

□

3. Define a linear transformation, $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, which maps the standard basis vectors, e_1 and e_2, in $\mathbb{R}^2$ to the vectors

$$w_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix},$$

respectively.

(a) Give the matrix representation, M_T, for T relative to the standard basis in $\mathbb{R}^2$.

***Answer**:* $M_T = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$. □

(b) Compute $\det(T)$. Does T preserve orientation?

***Solution**:* $\det(T) = \det(M_T) = -4 + 3 = -1$.

Since $\det(T) < 0$, T reverses orientation. □

(c) Show that T is invertible and compute the inverse of T.

Solution: T is invertible because $\det(T) \neq 0$.

The inverse of T is given by $T^{-1}v = M_T^{-1}v$, for all $v \in \mathbb{R}^2$, where

$$M_T^{-1} = \frac{1}{-1}\begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

Thus,

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

or

$$T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ -x - 2y \end{pmatrix}, \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

□

(d) Verify that $\lambda = 1$ is an eigenvalue of T and compute the corresponding eigenspace.

Solution: We verify that $T(v) = v$ has nontrivial solutions by solving the system

$$(M_T - I)v = \mathbf{0}, \qquad \text{for } v \in \mathbb{R}^2. \tag{3}$$

We reduce the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right)$$

to

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right);$$

so that the system in (3) is equivalent to the equation

$$x + 3y = 0,$$

which has solutions

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}.$$

Hence, $\lambda = 1$ is an eigenvalue of T and the corresponding eigenspace is

$$E_T(1) = \text{span}\left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right\}.$$

□