Exam 2 (Part II)

Friday, May 3, 2013

Name: ________________________________

This is the out–of–class portion of Exam 1. It is a closed–book and closed–notes exam. Students are expected to work individually on these problems. You may not consult with anyone, or the online notes in the course's website, or any other website. Show all significant work and provide reasoning for all your assertions.
Write your name on this page and staple it to your solutions. Turn in your solutions at the start of class on **Monday, May 6, 2013.**

I have read and agree to these instructions. Signature: ____________________

1. Let $A = \begin{pmatrix} -1 & 2 & -1 \\ -6 & 7 & -4 \\ -6 & 6 & -4 \end{pmatrix}$.

   (a) Verify that A has three distinct eigenvalues, λ_1, λ_2, and λ_3; list them in increasing order: $\lambda_1 < \lambda_2 < \lambda_3$. Compute λ_1, λ_2, and λ_3, and find corresponding eigenvectors v_1, v_2 and v_3.

   (b) Let v_1, v_2 and v_3 be the eigenvectors of A computed in part (a). Explain why the set $\mathcal{B} = \{v_1, v_2, v_3\}$ forms a basis for $\mathbb{R}^3$.

   (c) Set $Q = [v_1 \quad v_2 \quad v_3]$; that is, Q is the matrix whose columns are the eigenvectors of A in the ordered basis $\mathcal{B}$. Explain why Q is invertible and compute Q^{-1}.

   (d) Define $J = Q^{-1}AQ$. Compute J. What do you discover?

2. Let u_1 and u_2 denote a unit vector in $\mathbb{R}^n$, for $n \geqslant 2$, that are orthogonal to each other; i.e., $\langle u_1, u_2 \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.
   Define $f \colon \mathbb{R}^n \to \mathbb{R}^n$ by $f(v) = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2$ for all $v \in \mathbb{R}^n$.

   (a) Verify that f is linear.

   (b) Give the image, $\mathcal{I}_f$, and null space, $\mathcal{N}_f$, of f, and compute $\dim(\mathcal{I}_f)$.

   (c) The Dimension Theorem for a linear transformations, $T \colon \mathbb{R}^n \to \mathbb{R}^m$, states that
   $$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$
   Use the Dimension Theorem to compute $\dim(\mathcal{N}_f)$.