Solutions to Part II of Exam 2

1. Let $A = \begin{pmatrix} -1 & 2 & -1 \\ -6 & 7 & -4 \\ -6 & 6 & -4 \end{pmatrix}$.

   (a) Verify that A has three distinct eigenvalues, λ_1, λ_2, and λ_3; list them in increasing order: $\lambda_1 < \lambda_2 < \lambda_3$. Compute λ_1, λ_2, and λ_3, and find corresponding eigenvectors v_1, v_2 and v_3.

   ***Solution**:* Compute

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 & -1 \\ -6 & 7-\lambda & -4 \\ -6 & 6 & -4-\lambda \end{vmatrix}$$

to get

$$\det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + \lambda - 2,$$

which can be factored into

$$\det(A - \lambda I) = -(\lambda^2 - 1)(\lambda - 2),$$

or

$$\det(A - \lambda I) = -(\lambda + 1)(\lambda - 1)(\lambda - 2). \tag{1}$$

It follows from (1) that A has three distinct eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = 2.$$

In order to find an eigenvector corresponding to λ_1, solve the system

$$(A - \lambda_1 I)v = \mathbf{0} \tag{2}$$

by reducing the augmented matrix

$$\begin{pmatrix} 0 & 2 & -1 & | & 0 \\ -6 & 8 & -4 & | & 0 \\ -6 & 6 & -3 & | & 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -1/2 & | & 0 \end{pmatrix}$$

by means of elementary row operations. It then follows that the system in (2) is equivalent to the system

$$\begin{cases} x_1 & & = & 0; \\ & x_2 - \frac{1}{2}x_3 & = & 0, \end{cases}$$

which has solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}.$$

It then follows that

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = -1$.

Similar calculations can be used to show that

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_2 = 1$, and

$$v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_3 = 2$. □

(b) Let v_1, v_2 and v_3 be the eigenvectors of A computed in part (a). Explain why the set $\mathcal{B} = \{v_1, v_2, v_3\}$ forms a basis for $\mathbb{R}^3$.

***Solution**:* The set $\mathcal{B} = \{v_1, v_2, v_3\}$ is linearly independent because v_1, v_2 and v_3 are eigenvectors of A corresponding to distinct eigenvalues. □

(c) Set $Q = [v_1 \quad v_2 \quad v_3]$; that is, Q is the matrix whose columns are the eigenvectors of A in the ordered basis $\mathcal{B}$. Explain why Q is invertible and compute Q^{-1}.

***Solution**:* The matrix $Q = [v_1 \quad v_2 \quad v_3]$ is invertible because the columns are linearly independent.

To compute Q^{-1}, perform elementary row operations on the augmented matrix

$$[\ Q \quad | \quad I\ ],$$

where I is the 3×3 identity matrix to get

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 & -1 & 2 \\ 0 & 1 & 0 & | & 3 & -2 & 1 \\ 0 & 0 & 1 & | & -2 & 2 & -1 \end{pmatrix},$$

so that

$$Q^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -2 & 1 \\ -2 & 2 & -1 \end{pmatrix}.$$

□

(d) Define $J = Q^{-1}AQ$. Compute J. What do you discover?

***Solution**:* Compute

$$\begin{aligned} J &= Q^{-1}AQ \\ &= \begin{pmatrix} 1 & -1 & 2 \\ 3 & -2 & 1 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \\ -6 & 7 & -4 \\ -6 & 6 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \end{aligned}$$

to get

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Thus, J is a diagonal matrix with the eigenvalues of A along the main diagonal; that is,

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

□

2. Let u_1 and u_2 denote a unit vector in $\mathbb{R}^n$, for $n \geqslant 2$, that are orthogonal to each other; i.e., $\langle u_1, u_2 \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. Define $f \colon \mathbb{R}^n \to \mathbb{R}^n$ by $f(v) = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2$ for all $v \in \mathbb{R}^n$.

   (a) Verify that f is linear.

***Solution**:* For $v, w \in \mathbb{R}^n$, compute

$$\begin{aligned} f(v+w) &= \langle v+w, u_1\rangle u_1 + \langle v+w, u_2\rangle u_2 \\ &= (\langle v, u_1\rangle + \langle w, u_1\rangle)u_1 + (\langle v, u_2\rangle + \langle w, u_2\rangle)u_2 \\ &= \langle v, u_1\rangle u_1 + \langle w, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \langle w, u_2\rangle u_2 \\ &= (\langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2) + (\langle w, u_1\rangle u_1 + \langle w, u_2\rangle u_2) \\ &= f(v) + f(w). \end{aligned}$$

Similarly, for a scalar c and $v \in \mathbb{R}^n$,

$$\begin{aligned} f(cv) &= \langle cv, u_1\rangle u_1 + \langle cv, u_2\rangle u_2 \\ &= c\langle v, u_1\rangle u_1 + c\langle v, u_2\rangle u_2 \\ &= c(\langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2) \\ &= cf(v). \end{aligned}$$

□

(b) Give the image, $\mathcal{I}_f$, and null space, $\mathcal{N}_f$, of f, and compute $\dim(\mathcal{I}_f)$.

***Solution**:* The image of f is the set

$$\mathcal{I}_f = \{w \in \mathbb{R}^n \mid w = f(v) \text{ for some } v \in \mathbb{R}^n\}.$$

We claim that $\mathcal{I}_f = \text{span}\{u_1, u_2\}$. To see why this is so, first observe that

$$f(u_1) = \langle u_1, u_1\rangle u_1 + \langle u_1, u_2\rangle u_2 = \|u_1\|^2 u_1 = u_1, \tag{3}$$

since u_1 is a unit vector that is orthogonal to u_2. Similarly,

$$f(u_2) = u_2 \tag{4}$$

Let $w \in \text{span}\{u_1, u_2\}$; then $w = c_1u_1 + c_2u_2$, for some scalars c_1 and c_2. Now, by virtue of (3) and (4) the linearity of f,

$$w = c_1u_1 + c_2u_2 = c_1f(u_1) + c_2f(u_2) = f(c_1u_1 + c_2u_2),$$

which shows that $w \in \mathcal{I}_f$. We have therefore established that

$$w \in \text{span}\{u_1, u_2\} \Rightarrow w \in \mathcal{I}_f;$$

that is,

$$\text{span}\{u_1, u_2\} \subseteq \mathcal{I}_f. \tag{5}$$

Next, suppose that $w \in \mathcal{I}_f$; then, $w = f(v)$ for some $v \in \mathbb{R}^n$, so that

$$w = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 \in \text{span}\{u\}.$$

Thus,

$$\mathcal{I}_f \subseteq \text{span}\{u_1, u_2\}. \tag{6}$$

Combining (5) and (6) yields that

$$\mathcal{I}_f = \text{span}\{u_1, u_2\}.$$

Now, since u_1 and u_2 are orthogonal, they are linearly independent. It then follows that

$$\dim(\mathcal{I}_f) = 2. \tag{7}$$

The null space of f is the set

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid f(v) = \mathbf{0}\}.$$

Thus,

$$\begin{aligned} v \in \mathcal{N}_f \quad & \text{iff} \quad \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 = \mathbf{0} \\ & \text{iff} \quad \langle v, u_1\rangle = 0 \text{ and } \langle v, u_2\rangle = 0, \end{aligned}$$

since the set $\{u_1, u_2\}$ is linearly independent. It then follows that

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle v, u_1\rangle = 0 \text{ and } \langle v, u_2\rangle = 0\};$$

that is, $\mathcal{N}_f$ is the space of vectors which are orthogonal to u_1 and u_2. □

(c) The Dimension Theorem for a linear transformations, $T\colon \mathbb{R}^n \to \mathbb{R}^m$, states that

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$

Use the Dimension Theorem to compute $\dim(\mathcal{N}_f)$.

Solution: Using the dimension theorem and (7) we get that

$$\dim(\mathcal{N}_f) + 2 = n,$$

which implies that

$$\dim(\mathcal{N}_f) = n - 2.$$

□