Review Problems for Final Exam

1. Let W be a subspace of $\mathbb{R}^n$. Prove that $\text{span}(W) = W$.

2. Let S be linearly independent subset of $\mathbb{R}^n$. Suppose that $v \notin \text{span}(S)$. Show that the set $S \cup \{v\}$ is linearly independent.

3. Let W be a subspace of $\mathbb{R}^n$ with dimension $k < n$. Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W. Prove that there exist vectors $v_1, v_2, \ldots, v_{n-k}$ in $\mathbb{R}^n$ such that the set $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_{n-k}\}$ is a basis for $\mathbb{R}^n$.

4. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Prove that if $Ax = b$ has a solution x in $\mathbb{R}^n$, then $\langle b, v\rangle = 0$ for every v is the null space of A^T.

5. Let $A \in \mathbb{M}(m, n)$ and write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ denote the rows of A. Define $\mathcal{R}_A^\perp$ to be the set

$$\mathcal{R}_A^\perp = \{w \in \mathbb{R}^n \mid R_i w = 0 \text{ for all } i = 1, 2, \ldots, m\};$$

that is, $\mathcal{R}_A^\perp$ is the set of vectors in $\mathbb{R}^n$ which are orthogonal to the vectors $R_1^T, R_2^T, \ldots, R_m^T$ in $\mathbb{R}^n$.

   (a) Prove that $\mathcal{R}_A^\perp$ is a subspace of $\mathbb{R}^n$.

   (b) Prove that $\mathcal{R}_A^\perp = \mathcal{N}_A$.

   (c) Let v denote a vector in $\mathbb{R}^n$. Prove that if $v \in \mathcal{N}_A$ and $v^T \in \mathcal{R}_A$, then $v = \mathbf{0}$.

6. Let B be an $n \times n$ matrix satisfying $B^3 = 0$ and put $A = I + B$, where I denotes the $n \times n$ identity matrix. Prove that A is invertible and compute A^{-1} in terms of I, B and B^2.

7. Let $A, B \in \mathbb{M}(n, n)$. Show that $\det(AB) = \det(BA)$.

8. Given an $n \times n$ matrix $A = [a_{ij}]$, the trace of A, denoted $\text{tr}(A)$, is the sum of the entries along the main diagonal of A; that is $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

   Let A and B denote $n \times n$ matrices. Show that $\text{tr}(AB) = \text{tr}(BA)$.

9. Let A and B be $n \times n$ matrices such that $B = Q^{-1}AQ$ for some invertible $n \times n$ matrix Q.

   Prove that A and B have the same determinant and the same trace.

10. Let $A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$.

    (a) Find a basis $\mathcal{B} = \{v_1, v_2\}$ for $\mathbb{R}^2$ made up of eigenvectors of A.

    (b) Let Q be the 2×2 matrix $Q = [\, v_1 \quad v_2 \,]$, where $\{v_1, v_2\}$ is the basis of eigenvectors found in (a) above. Verify that Q is invertible and compute $Q^{-1}AQ$.

    (c) Use the result in part (b) above to find a formula for for computing A^k for every positive integer k. Can you say anything about $\lim_{k\to\infty} A^k$?

11. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ denote a linear transformation. Let W denote the null space, $\mathcal{N}_T$, of T. Assume that W has dimension $k < n$. Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W and $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_{n-k}\}$ be a basis for $\mathbb{R}^n$. Prove that that the set $\{T(v_1), T(v_2), \ldots, T(v_{n-k})\}$ is a basis for $\mathcal{I}_T$, the image of T. Deduce that
$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$

12. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ denote a linear transformation. Prove that if λ is an eigenvalue of T, then λ^k is an eigenvalue of T^k for every positive integer k. If μ is an eigenvalue of T^k, is $\mu^{1/k}$ always and eigenvalue of T?

13. Let $\mathcal{E} = \{e_1, e_2\}$ denote the standard basis in $\mathbb{R}^2$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function satisfying: $f(e_1) = e_1 + e_2$ and $f(e_2) = 2e_1 - e_2$.

    Give the matrix representations for f and $f \circ f$ relative to $\mathcal{E}$.

14. A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows: Each vector $v \in \mathbb{R}^2$ is reflected across the y–axis, and then doubled in length to yield $f(v)$.

    Verify that f is linear and determine the matrix representation, M_f, for f relative to the standard basis in $\mathbb{R}^2$.

15. Find a 2×2 matrix A such that the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(v) = Av$ maps the coordinates of any vector, relative to the standard basis in $\mathbb{R}^2$, to its coordinates relative the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.