## Assignment #2

**Due on Friday, January 31, 2014**

**Read** Sections 2.1 and 2.3 on $\sigma$–*fields* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

**Read** Sections 2.2 on *Some Set Algebra* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

**Read** Section 1.4 on *Set Theory* in DeGroot and Schervish.

**Background and Definitions**

- A $\sigma$-field, $\mathcal{B}$, is a collection of subsets of a sample space $\mathcal{C}$, referred to as **events**, which satisfy:
  (1) $\emptyset \in \mathcal{B}$ ($\emptyset$ denotes the empty set)
  (2) If $E \in \mathcal{B}$, then its complement, $E^c$, is also an element of $\mathcal{B}$.
  (3) If $(E_1, E_2, E_3 \ldots)$ is a sequence of events, then
$$E_1 \cup E_2 \cup E_3 \cup \ldots = \bigcup_{k=1}^{\infty} E_k \in \mathcal{B}.$$
- Let $\mathcal{S}$ denote a collection of subsets of a sample space $\mathcal{C}$. The $\sigma$–field generated by $\mathcal{S}$, denoted by $\mathcal{B}(\mathcal{S})$, is the smallest $\sigma$–field in $\mathcal{C}$ which contains $\mathcal{S}$.
- $\mathcal{B}_o$ denotes the Borel $\sigma$–field of the real line, $\mathbb{R}$. This is the $\sigma$–field generated by the semi–infinite intervals
$$(-\infty, b], \qquad \text{for } b \in \mathbb{R}.$$

**Do** the following problems

1. Let $A$, $B$ and $C$ be subsets of a sample space $\mathcal{C}$. Prove the following
   (a) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
   (b) If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

2. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a $\sigma$–field of subsets of $\mathcal{C}$. Prove that if $\{E_1, E_2, E_3 \ldots\}$ is a sequence of events in $\mathcal{B}$, then

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.$$

*Hint:* Use De Morgan's Laws.

3. Let $\mathcal{C}$ be a sample space and $\mathcal{B}$ be a $\sigma$–field of subsets of $\mathcal{C}$. For fixed $B \in \mathcal{B}$ define the collection of subsets

$$\mathcal{B}_B = \{D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$

Show that $\mathcal{B}_B$ is a $\sigma$–field.

*Note:* In this case, the complement of $D \in \mathcal{B}_B$ has to be understood as $B \backslash D$; that is, the complement relative to $B$. The $\sigma$–field $\mathcal{B}_B$ is the $\sigma$–field $\mathcal{B}$ restricted to $B$, or *conditioned on* $B$.

4. Let $\mathcal{S}$ denote the collection of all bounded, open intervals $(a, b)$, where $a$ and $b$ are real numbers with $a < b$. Show that

$$\mathcal{B}(\mathcal{S}) = \mathcal{B}_o;$$

that is, the $\sigma$–field generated by bounded open intervals is the Borel $\sigma$–field.

*Hints:*

- We have already seen in the lecture that $\mathcal{B}_o$ contains all bounded open intervals.
- Observe also that the semi–infinite open interval $(b, \infty)$ can be expressed as the union of the sequence of bounded intervals $(b, k)$, for $k = 1, 2, 3, \ldots$

5. Show that for every real number $a$, the singleton $\{a\}$ is in the Borel $\sigma$–field $\mathcal{B}_o$.

*Hint:* Express $\{a\}$ as an intersection of a sequence of open intervals.