Review Problems for Exam 2

(1) Let X and Y be independent Exponential(1) random variables. Put $Z = \dfrac{Y}{X}$. Compute the distribution functions $F_Z(z)$ and $f_Z(z)$.

(2) A random point (X, Y) is distributed uniformly on the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

(a) Give the joint pdf for X and Y.

(b) Compute the following probabilities: (i) $P(X^2 + Y^2 < 1)$, (ii) $P(2X - Y > 0)$, (iii) $P(|X + Y| < 2)$.

(3) Let $F_{(X,Y)}$ be the joint cdf of two random variables X and Y. For real constants $a < b$, $c < d$, show that

$$\Pr(a < X \leq b, c < Y \leq d) = F_{(X,Y)}(b,d) - F_{(X,Y)}(b,c) - F_{(X,Y)}(a,d) + F_{(X,Y)}(a,c).$$

Use this result to show that

$$F(x,y) = \begin{cases} 1 & \text{if } x + 2y \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

cannot be the joint cdf of two random variables.

(4) The random pair (X, Y) has the joint distribution

X \ Y	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

(a) Show that X and Y are not independent.

(b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

(5) Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses of a fair coin. Are X and Y independent random variables?

(6) Prove that if the joint cdf of X and Y satisfies

$$F_{X,Y}(x,y) = F_X(x)F_Y(y),$$

then for any pair of intervals (a,b) and (c,d),

$$P(a < X \leq b, c < Y \leq d) = P(a < X \leq b)P(c < Y \leq d).$$

(7) Let $g(t)$ denote a non–negative, integrable function of a single variable with the property that

$$\int_0^\infty g(t)\ dt = 1.$$

Define

$$f(x,y) = \begin{cases} \dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}} & \text{for } 0 < x < \infty,\ 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f(x,y)$ is a joint pdf for two random variables X and Y.

(8) Suppose that X and Y are independent random variables such that $X \sim \text{Uniform}(0,1)$ and $Y \sim \text{Exponential}(1)$.

(a) Let $Z = X + Y$. Find F_Z and f_Z.

(b) Let $U = Y/X$. Find F_U and f_U.

(9) Let $X \sim \text{Exponential}(1)$, and define Y to be the integer part of $X+1$; that is, $Y = i+1$ if and only if $i \leq X < i+1$, for $i = 0, 1, 2, \ldots$ Find the pmf of Y, and deduce that $Y \sim \text{Geometric}(p)$ for some $0 < p < 1$. What is the value of p?

(10) Suppose that two persons make an appointment to meet between 5 PM and 6 PM at a certain location and they agree that neither person will wait more than 10 minutes for each person. If they arrive independently at random times between 5 PM and 6 PM, what is the probability that they will meet?