Assignment #1

Due on Wednesday, January 29, 2014

Read Section 2.1 on *Modeling Fluid Flow* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read pages 1–4 in the text.

Do the following problems

1. Let R denote an open subset of $\mathbb{R}^3$ and $f\colon R \to \mathbb{R}$ denote a continuous function. Suppose that
$$\iiint_B f\ dV = 0$$
for all bounded subsets, B, of R with smooth boundary. Show that $f(x, y, z) = 0$ for all $(x, y, z) \in R$.

2. Let R be as in Problem 1 and $\overrightarrow{F}\colon R \to \mathbb{R}^3$ denote a continuous vector field. Suppose that
$$\iiint_B \overrightarrow{F}\ dV = 0, \tag{1}$$
for all bounded subsets, B, of R with smooth boundary, where the 0 on the right–hand side of (1) denotes the zero vector in $\mathbb{R}^3$. Show that $\overrightarrow{F}(x, y, z) = 0$ for all $(x, y, z) \in R$.

3. Let R denote an open subset of $\mathbb{R}^3$ and $f\colon R \to \mathbb{R}$ denote a C^1 function. Let B denote a bounded subset of R with smooth boundary ∂B. Apply the divergence theorem to show that
$$\iiint_B \frac{\partial f}{\partial x}\ dV = \iint_{\partial B} f n_1\ dA, \tag{2}$$
where n_1 is the first component of the outward unit normal, $\vec{n}$, to the boundary of B.

   Write analogous expressions to that in (2) involving $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$.

4. In the lecture notes we derived the one–dimensional continuity equation,
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3}$$

for x in some interval $I \subseteq \mathbb{R}$ and for all $t \in \mathbb{R}$. In (3) we are assuming that u and ρ are C^1 functions of (x,t), for $x \in I$ and $t \in \mathbb{R}$.

For the special case in which u is constant (say, $u(x,t) = c$ for all $x \in I$ and all t), (3) becomes

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0, \quad x \in I, t \in \mathbb{R}. \tag{4}$$

Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function. Verify that

$$\rho(x,t) = f(x - ct), \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

solves the PDE in (4) for $I = \mathbb{R}$.

5. Suppose that u and ρ are solutions to the system of PDEs

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) = 0, & x \in \mathbb{R}, t \in \mathbb{R}; \\ \\ \dfrac{\partial u}{\partial x} = 0, & x \in \mathbb{R}, t \in \mathbb{R} \end{cases} \tag{5}$$

   (a) Verify that ρ solves the PDE

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}. \tag{6}$$

   (b) Assume that u is known, ρ solves (6) and that $x = x(t)$ is a solution to the ordinary differential equation

$$\frac{dx}{dt} = u(x,t), \quad t \in \mathbb{R}.$$

   Compute $\dfrac{d}{dt}[\rho(x(t),t)]$, for all $t \in \mathbb{R}$. What do you conclude?