Assignment #2

Due on Friday, January 31, 2014

Read Section 2.1 on *Modeling Fluid Flow* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions

- **Pathlines**. For a fluid of C^1 density ρ flowing in a region R of $\mathbb{R}^3$ according to a C^1 velocity field $\vec{u} = (u_1, u_2, u_3)$, the pathlines are solutions to the system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = u_1(x(t), y(t), z(t), t); \\[2ex] \dfrac{dy}{dt} = u_2(x(t), y(t), z(t), t); \\[2ex] \dfrac{dz}{dt} = u_3(x(t), y(t), z(t), t), \end{cases} \tag{1}$$

- **Material Derivative**. Given a C^1 scalar field, g, the time derivative of g along the pathlines,

$$\begin{aligned} \frac{d}{dt}[g(x(t), y(t), z(t), t)] &= \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial g}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial g}{\partial t} \\ &= u_1\frac{\partial g}{\partial x} + u_2\frac{\partial g}{\partial y} + u_3\frac{\partial g}{\partial z} + \frac{\partial g}{\partial t}, \end{aligned} \tag{2}$$

  is called the **material derivative** of g, and is denoted by $\dfrac{Dg}{Dt}$; so that

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \vec{u}\cdot\nabla g, \tag{3}$$

  where $\nabla g = \left(\dfrac{\partial g}{\partial x}, \dfrac{\partial g}{\partial y}, \dfrac{\partial g}{\partial z}\right)$ is the gradient of g.

The material derivative of a C^1 vector field $\overrightarrow{G} = (g_1, g_2, g_3)$, is

$$\frac{D\overrightarrow{G}}{Dt} = \left(\frac{Dg_1}{Dt}, \frac{Dg_2}{Dt}, \frac{Dg_3}{Dt}\right), \tag{4}$$

which can be written as

$$\frac{D\overrightarrow{G}}{Dt} = \frac{\partial \overrightarrow{G}}{\partial t} + \vec{u} \cdot \nabla \overrightarrow{G}. \tag{5}$$

Do the following problems

1. Let f and g denote C^1 scalar fields defined in R. Use the definition of the material derivative in (2) and (3) to verify that

$$\frac{D}{Dt}[fg] = f\frac{Dg}{Dt} + g\frac{Df}{Dt}.$$

2. Let f denote a C^1 scalar field and $\overrightarrow{G}$ a C^1 vector field defined in R. Use the definition of the material derivative in (4) and (5), and the result in Problem 1 to derive an expression for $\dfrac{D}{Dt}[f\overrightarrow{G}]$.

3. Compute $\dfrac{D\vec{u}}{Dt}$.

4. Let $\overrightarrow{F}$ and $\overrightarrow{G}$ denote C^1 vector fields in R. Compute

$$\frac{d}{dt}[\overrightarrow{F}(x(t), y(t), z(t), t) \cdot \overrightarrow{G}(x(t), y(t), z(t), t)],$$

and use your result to derive and expression for $\dfrac{D}{Dt}[\overrightarrow{F} \cdot \overrightarrow{G}]$.

5. Compute $\dfrac{D}{Dt}[\|\vec{u}\|^2]$, where $\|\vec{u}\|$ denotes the Euclidean norm of $\vec{u}$.