Assignment #3

Due on Monday, February 3, 2014

Read Section 2.1 on *Modeling Fluid Flow* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

- **Flow Map**. For a fluid of C^1 density ρ flowing in a region R of $\mathbb{R}^3$ according to a C^1 velocity field $\vec{u} = (u_1, u_2, u_3)$, solutions to the system of ordinary differential equations,

$$\begin{cases} \dfrac{dx}{dt} & = & u_1(x(t), y(t), z(t), t); \\ \\ \dfrac{dy}{dt} & = & u_2(x(t), y(t), z(t), t); \\ \\ \dfrac{dz}{dt} & = & u_3(x(t), y(t), z(t), t), \end{cases} \tag{1}$$

give rise to the flow map $\varphi_t \colon R \to R$ as follows: The map

$$t \mapsto \varphi_t(x, y, z),$$

for t in some maximal interval of existence containing 0, is the unique solution to the system in (**??**) subject to the initial conditions

$$\begin{cases} x(0) & = & x; \\ y(0) & = & y; \\ z(0) & = & z. \end{cases}$$

The map $\varphi_t \colon R \to R$ is C^1 and invertible.

- **The Transport Theorem**. Let f denote a C^1 scalar field defined in a region R in space in which a fluid with velocity field $\vec{u}$ is flowing. Let B be any open bounded subset of R and define $B_t = \varphi_t(B)$, where φ_t is the flow map. The Transport theorem states that

$$\frac{d}{dt} \iiint_{B_t} f \, dV = \iiint_{B_t} \left(\frac{\partial f}{\partial t} + \nabla \cdot (f\vec{u}) \right) \, dV \tag{2}$$

- **Incompressible Flow**. The flow associated with a C^1 velocity field $\vec{u}$ defined in a region R is said to be incompressible if $\vec{u}$ satisfies the PDE

$$\nabla \cdot \vec{u} = 0, \qquad \text{in } R.$$

Do the following problems

1. Derive the Transport Theorem in (**??**).

   Suggestion: Use the change of variables provided by the flow map, φ_t, to write
   $$\iiint_{B_t} f\ dV = \iiint_{B} fJ\ dxdydz,$$
   where J is the Jacobian of the flow map, and use the fact that
   $$\frac{\partial J}{\partial t} = (\nabla \cdot \vec{u})J.$$

2. Let B be an open bounded subset of a region R in space and set $B_t = \varphi_t(B)$, where φ_t is the flow map associated with a velocity field $\vec{u}$. Let $v(t)$ denote the volume of B_t for each t.

   (a) Use the Transport Theorem to derive an expression for computing $\dfrac{dv}{dt}$.

   (b) Show that, for incompressible flow, the volume of B_t remains constant for all t.

3. Write the conservation of mass PDE for incompressible flow.

4. Write the conservation of momentum PDE for an incompressible, ideal fluid.

5. Let f denote a C^2 scalar field defined in an open region, R, in $\mathbb{R}^3$, and put $\overrightarrow{F} = \nabla f$; that is, define the vector field $\overrightarrow{F}$ to be the gradient of the scalar field f. Use the divergence theorem to derive the the expression
   $$\iiint_{B} \Delta f\ dV = \iint_{\partial B} \frac{\partial f}{\partial n}\ dA,$$
   for any bounded subset, B, of R with smooth boundary, where
   $$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
   is called the Laplacian of f, and
   $$\frac{\partial f}{\partial n} = \nabla f \cdot \vec{n}$$
   is the directional derivative of f along the boundary of B in the direction of the outward unit normal.