Assignment #8

Due on Wednesday, February 19, 2014

Read Section 2.3.3 on the *Vibrating String* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read pages 1–13 in the text.

Do the following problems

1. Let R denote an open subset of $\mathbb{R}^3$ with smooth boundary, ∂R, and $f\colon R \to \mathbb{R}$ and $\varphi\colon R \to \mathbb{R}$ denote C^1 functions. Use the result of Problem 3 in Assignment #1 to derive the following integration by parts formula:

$$\iiint_R f\frac{\partial \varphi}{\partial x}\, dV = \iint_{\partial R} f\varphi n_1\, dA - \iiint_R \frac{\partial f}{\partial x}\varphi\, dV, \qquad (1)$$

   where n_1 is the first component of the outward unit normal, $\vec{n}$, to the boundary of B.

   Write analogous expressions to that in (1) involving partial derivatives with respect to y and with respect to z, respectively.

   Obtain analogous result for an open region, R, in $\mathbb{R}^2$.

2. In class, and in the lecture notes, we derived the one dimensional wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \tau\frac{\partial^2 u}{\partial x^2} = 0, \quad \text{for } 0 < x < L, t > 0, \qquad (2)$$

   that determines small amplitude vibrations of a string of length L, linear density ρ, and constant tension τ, that is fixed at the end–points $x = 0$ and $x = L$.

   The total energy (kinetic plus potential) at time t of the string is given by

$$E(t) = \frac{1}{2}\int_0^L \rho u_t^2\, dx + \frac{1}{2}\int_0^L \tau u_x^2\, dx.$$

   Assume that u is a C^2 solution of the PDE in (2). Compute the rate of change of total energy, $\dfrac{dE}{dt}$. What do you conclude about E?

3. Let w be a C^2 solution to the initial–boundary value problem

$$\begin{cases} \rho\dfrac{\partial^2 w}{\partial t^2} - \tau\dfrac{\partial^2 w}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ \\ w(x,0) = 0, & \text{for all } x \in [0,L]; \\ w_t(x,0) = 0, & \text{for all } x \in [0,L]; \\ w(0,t) = w(L,t) = 0, & \text{for all } t. \end{cases} \tag{3}$$

   Show that w must be the 0 function.

   Suggestion: Define

$$E(t) = \frac{1}{2}\int_0^L \rho w_t^2\; dx + \frac{1}{2}\int_0^L \tau w_x^2\; dx, \qquad \text{for all } t,$$

   and use the result of Problem 2.

4. Prove that the initial–boundary value problem

$$\begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2} - \tau\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ \\ u(x,0) = f(x), & \text{for all } x \in [0,L]; \\ u_t(x,0) = g(x), & \text{for all } x \in [0,L]; \\ u(0,t) = u(L,t) = 0, & \text{for all } t, \end{cases} \tag{4}$$

   where f and g are given continuous functions defined in $[0,L]$, can have at most one solution.

   Suggestion: Use the result of Problem 3.

5. Let f and g denote twice–differentiable, real valued functions of a single variable and define

$$u(x,t) = f(x+ct) + g(x-ct), \qquad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}.$$

   Show that u solves that one–dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$