Notes on Partial Differential Equations

Preliminary Lecture Notes

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Contents

Chapter 1

Preface

This course is an introduction to the theory and applications of partial differential equations (PDEs). PDEs are expressions involving functions of several variables and its derivatives in which we seek to find one of the functions, or a set of functions, subject to some initial conditions (if time is involved as one of the variables) or boundary conditions. They arise naturally when modeling physical or biological systems in which assumptions of continuity and differentiability are made about the quantities in question. In these notes we will discuss several modeling situations that give rise to PDEs.

In problems involving PDEs we are mainly interested in the question of existence of solutions. In a few cases, answering these questions amounts to coming up with formulas for the solutions. In these notes we will discuss a few techniques for constructing solutions (e.g., separation of variables, series expansions and Green's function methods) for the special case of linear equations. In most cases, however, explicit constructions of solutions are not possible. In these cases, the only recourse we have is analytical proofs of existence, or nonexistence, and qualitative analysis to deduce properties of solutions. Once an existence theorem is obtained for a particular PDE problem, the next step in the analysis might involve approximation techniques to get information on the behavior and property of solutions.

The field of PDEs is vast and complex. The complexity is derived from the great diversity of types of PDEs and of techniques for approaching their analysis. PDEs range from linear to nonlinear; single equations to systems; and from first degree to higher degree. There is also a further classification determined by the behavior of solutions of certain classes of equations. Over the years researchers have identified three major classes of PDEs: hyperbolic, elliptic and parabolic. Archetypal instances of these classes of PDEs are the classical equations of mathematical physics: the wave equation, Laplace's or Poisson' equations, and the heat or diffusion equations, respectively. In these notes we will provide examples of analysis for each of these types of equations. Added to the complexity of the field of the PDEs is the fact that many problems can be of mixed type. Hence, ability to recognize types of PDEs, or how a given problem can change from one type to the other, is very important in the analysis of problems involving PDEs.

Finally, some classes of PDE problems have a particular structure that lends itself to certain kind of general approach for analysis. For instance, many problems in PDEs can be can be formulated in terms of finding a function for which a certain quantity is minimized, or maximized, over a class of functions. The problem of finding such an optimizer is known as a variational problem. In these notes we present an introduction to variational techniques for solving a class of PDE problems that are amenable to the variational treatment.

Chapter 2

How Do PDEs Arise?

In general, a partial differential equation for a function, u, of several variables, $u(x_1, x_2, \ldots, x_n)$, is an expression of the form

$$F(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_n x_n}, \ldots) = 0, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)$ and $u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_n x_n}, \ldots$ denote partial derivatives of u, for some function, F, of several variables. For example, in the simplest case in which u is a function of time, $t \in \mathbb{R}$, and a single space variable $x \in \mathbb{R}$, an instance of (2.1) is provided by

$$u_t - k u_{xx} = 0, \tag{2.2}$$

for some constant k.

While we are usually interested in knowing when equations like (2.1) and (2.2) have solutions subject to some initial and/or boundary conditions, in this chapter we will focus on the questions of how those equations arise in practice. For instance, the equation in (2.2) describes one–dimensional heat flow ($u(x,t)$ in this case denotes the temperature at time t and location x), or one–dimensional diffusion ($u(x,t)$ denotes the concentration of a substance at time t and location x). We begin by deriving a system of PDEs that describe the motion of fluids.

2.1 Modeling Fluid Flow

In this section we illustrate the use of a very important modeling principle, which we shall refer to as a **conservation principle**. This is a rather general principle that can be applied in situations in which the evolution in time of the quantity of a certain entity within a certain system is studied. For instance, suppose the quantity of a certain substance confined within a system is given by a continuous function of time, t, and is denoted by $Q(t)$ (the assumption of continuity is one that needs to be justified by the situation at hand). A conservation principle states that the rate at which a the quantity $Q(t)$ changes has to be accounted by how much of the substance goes into the system and how much of it goes out of the system. For the case in which Q is also assumed to be differentiable (again, this is a mathematical assumption that would need some justification), the conservation principle can be succinctly stated as

$$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out.} \tag{2.3}$$

In the cases to be considered in this section, the conservation principle in (2.3) might lead to a differential equation, or a system of differential equations, and so the theory of differential equations will be used to help in the analysis of the model.

In the derivation of the equations governing fluid motion, we will have the opportunity to apply the conservation principle in (2.3) several times.

Suppose we are following the motion of a fluid in some region R in three–dimensional space; see Figure 2.1.1. We assume that the fluid is a continuum

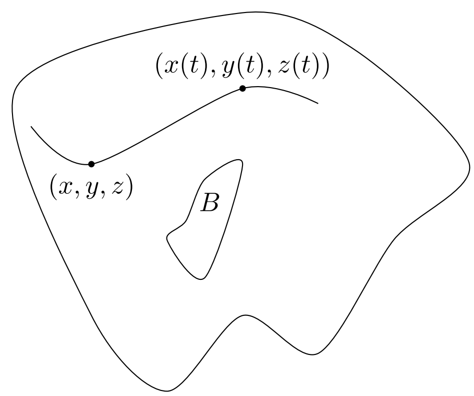

Figure 2.1.1: Region R

with density function $\rho(x, y, x, t)$, in units of mass per unit volume, so that the mass of a fluid element of volume $dV = dxdydz$ around a point (x, y, z) at time t is, approximately,

$$\rho(x, y, x, t)dV,$$

where dV denotes the volume of the fluid element. It then follows that the mass of fluid contained in a subregion B of R (see Figure 2.1.1) at time t is given by

$$M(B, t) = \iiint_B \rho(x, y, x, t)\, dV. \tag{2.4}$$

We assume throughout this discussion that ρ is a continuous function.

We also assume that each fluid element located at (x, y, z) at time t moves according to a velocity vector $\vec{u} = (u_1, u_2, u_3)$, where u_1, u_2 and u_3 are differentiable functions of (x, y, z, t). Thus, the path that a fluid element located at (x, y, z) at time $t = 0$ will follow is determined by the following system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} &= u_1(x(t), y(t), z(t), t); \\[2ex] \dfrac{dy}{dt} &= u_2(x(t), y(t), z(t), t); \\[2ex] \dfrac{dz}{dt} &= u_3(x(t), y(t), z(t), t), \end{cases} \tag{2.5}$$

subject to the initial conditions

$$\begin{cases} x(0) &= x; \\ y(0) &= y; \\ z(0) &= z. \end{cases} \tag{2.6}$$

If we assume that the components of the velocity field $\vec{u}$ are differentiable with continuous derivatives throughout the region R and for all times t (i.e., $\vec{u}$ is a C^1 vector field), then a solution to the system of ordinary differential equations in (2.5) subject to the initial conditions in (2.6) is guaranteed to exist over some maximal interval of time containing 0. The solution $(x(t), y(t), y(t))$ of the system in (2.5) subject to the initial conditions in (2.6) defines a path in space,

$$t \mapsto (x(t), y(t), y(t)),$$

for t in the maximal interval of existence, which describes the motion of a fluid element located at (x, y, z) at time $t = 0$. The path traced by the fluid element as it moves in time is called a **pathline**; Figure 2.1.1 shows what a pathline through (x, y, z) might look like. If we knew the velocity field at any point in space and at any time, we could compute the pathline through (x, y, z) by integrating the equations in (2.5) and imposing the initial conditions in (2.6):

$$\begin{aligned} x(t) &= x + \int_0^t u_1(x(\tau), y(\tau), z(\tau), \tau)\, d\tau; \\ y(t) &= y + \int_0^t u_2(x(\tau), y(\tau), z(\tau), \tau)\, d\tau; \\ z(t) &= z + \int_0^t u_3(x(\tau), y(\tau), z(\tau), \tau)\, d\tau. \end{aligned} \tag{2.7}$$

However, the velocity field is usually not known, and we need to do more modeling to find equations involving u_1, u_2 and u_3 that we hope we can solve.

2.1.1 The Continuity Equation

Consider a subregion, B, of R, with smooth boundary ∂B, as that pictured in Figure 2.1.1. The mass of the fluid contained at time t in that region, $M_B(t)$, is given by equation (2.4),

$$M_B(t) = \iiint_B \rho(x, y, x, t)\ dV. \tag{2.8}$$

By the principle of conservation of mass, the rate of change in the mass of fluid contained in B has to be accounted for by how much fluid is entering the region and how much is leaving per unit of time:

$$\frac{dM_B}{dt} = \text{ Rate of fluid into } B - \text{ Rate of fluid out of } B. \tag{2.9}$$

The equation in (2.9) is an instance of the conservation principle in (2.3).

If we assume that ρ is a C^1 function in R, we can compute the left–hand side of the equation by differentiating under the integral in (2.8):

$$\frac{dM_B}{dt} = \iiint_B \frac{\partial \rho}{\partial t}(x, y, x, t)\ dV. \tag{2.10}$$

Next, we compute the right–hand side of the expression in (2.9). Let $\vec{n}$ denote the unit vector normal to the boundary, ∂B, of the region B pointing outward. The outward unit normal, $\vec{n}(x, y, z)$, to the boundary of B is guaranteed to exist at every point $(x, y, z) \in \partial B$ if we assume that ∂B is a smooth surface. Then, the rate of fluid passing through an element of area, dA, on the surface ∂B can be expressed, approximately, as

$$\rho\ \vec{u} \cdot \vec{n}\ dA, \tag{2.11}$$

where $\vec{u} \cdot \vec{n}$ denotes the dot product of $\vec{u}$ and $\vec{n}$. Note that the expression in (2.11) is in units of mass per unit of time. Integrating the expression in (2.11) over the boundary of B yields the net **flux** of mass across the surface ∂B,

$$\iint_{\partial B} \rho\ \vec{u} \cdot \vec{n}\ dA. \tag{2.12}$$

Since the outward unit normal, $\vec{n}$, points away from the region B, the expression in (2.12) measures the flux of fluid away from the region B, if it is positive; if the expression in (2.12) is negative, it measures the net amount of fluid per unit time that enters B. We can therefore write the conservation principle in (2.9) as

$$\frac{dM_B}{dt} = -\iint_{\partial B} \rho\ \vec{u} \cdot \vec{n}\ dA. \tag{2.13}$$

To understand the reason for the minus sign on the right–hand side of the expression in (2.13), observe that a net increase in the amount of fluid in the region B, which yields a positive sign for the derivative in the left–hand side of (2.13), corresponds to a net amount of fluid flowing into the region B across the boundary ∂B.

Since we are assuming that the boundary of B is smooth, we can apply the Divergence Theorem to rewrite the integral in the right–hand side of (2.13) as follows:

$$\iint_{\partial B} \rho\, \vec{u} \cdot \vec{n}\; dA = \iiint_B \nabla \cdot (\rho\vec{u})\; dV, \tag{2.14}$$

where $\nabla \cdot (\rho\vec{u})$ denotes the divergence of the vector field $\rho\vec{u}$; that is,

$$\nabla \cdot (\rho\vec{u}) = \frac{\partial}{\partial x}(\rho u_1) + \frac{\partial}{\partial y}(\rho u_2) + \frac{\partial}{\partial z}(\rho u_3). \tag{2.15}$$

In view of (2.10) and (2.14), we see that we can rewrite the conservation equation in (2.13) as

$$\iiint_B \frac{\partial \rho}{\partial t}\; dV = -\iiint_B \nabla \cdot (\rho\vec{u})\; dV,$$

or

$$\iiint_B \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u})\right] dV = 0. \tag{2.16}$$

If we assume that the vector field $\vec{u}$ and the scalar field ρ are C^1 functions over R and for all times t, then the fact that (2.16) holds true for any subregion B of R with smooth boundary implies that integrand on the left–hand side of (2.16) must be 0 over R and for all t; that is,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0, \quad \text{in } R \text{ and for all } t. \tag{2.17}$$

The equation in (2.17) is an example of a partial differential equation (PDE) involving the functions ρ, u_1, u_2 and u_3; in fact, using the definition of divergence (see (2.15)), the PDE in (2.17) can be rewritten as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_1) + \frac{\partial}{\partial y}(\rho u_2) + \frac{\partial}{\partial z}(\rho u_3) = 0. \tag{2.18}$$

The PDE in (2.17) is called the **continuity equation** and it expresses the conservation principle for a quantity of density ρ that flows according to a velocity field $\vec{u}$ in some region in space. For one–dimensional flow with linear density $\rho(x,t)$ and scalar velocity field $u(x,t)$, for $x \in \mathbb{R}$ and $t \in \mathbb{R}$, the continuity equation reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0; \tag{2.19}$$

see (2.18). The equation in (2.19) is an example of a first order PDE because the first derivatives of the functions ρ and u are involved. As it stands, the PDE in (2.19) involves two unknown functions, the density, ρ, and the velocity, u. Thus, we will need one more relation or equations in order for us to even begin to solve the problem posed by the modeling that led to the PDE in (2.19). An interesting example is provided by the following application to modeling traffic flow.

Example 2.1.1 (Modeling Traffic Flow)**.** Consider the unidirectional flow of traffic in a one–lane, straight road depicted in Figure 2.1.2. In this idealized road, vehicles are modeled by moving points. The location, x, of a point–vehicle is measured from some reference point along an axis parallel to the road. We

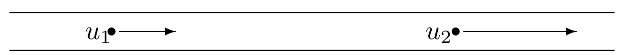

Figure 2.1.2: One–lane unidirectional flow

postulate a traffic density, $\rho(x,t)$, measured in units of number of cars per unit length of road at location x and time t. We interpret $\rho(x,t)$ as follows: Consider a section of the road from x to $x+\Delta x$ at time t. Let $\Delta N([x, x+\Delta x], t)$ denote the number of cars in the section $[x, x+\Delta x]$ at time t. We define $\rho(x,t)$ by the expression

$$\rho(x,t) = \lim_{\Delta x \to 0} \frac{\Delta N([x, x+\Delta x], t)}{\Delta x}, \tag{2.20}$$

provided that the limit on the right–hand side of (2.20) exists. It follows from (2.20) that, if a continuous traffic density, $\rho(x,t)$, is known for all x and t, then the number of cars in a section of the road from $x = a$ to $x = b$, where $a < b$, at time t is given by

$$\Delta N([a,b], t) = \int_a^b \rho(x,t)\ dx.$$

We assume that at each point x along the road and at each time t the velocity of vehicle at that location and time is dictated by a function $u(x,t)$, which we also assume to be a C^1 function. It follows from these assumptions and the derivations in this section that the one–dimensional equation of continuity in (2.19) applies to this situation.

Ideally, we would like to find a solution, ρ, to (2.19) subject to some initial condition

$$\rho(x,0) = \rho_o(x), \tag{2.21}$$

for some initial traffic density profile, ρ_o, along the road. In order to solve this problem, we postulate that u is a function of traffic density—the higher the density, the lower the traffic speed, for example. We may therefore write

$$u = f(\rho, \Lambda), \tag{2.22}$$

where f is a continuous function of ρ and a set of parameters, Λ. Some of the parameters might be a maximum density, $\rho_{\max}$, dictated by bumper to bumper traffic, and a maximum speed, $v_{\max}$; for instance, $v_{\max}$ is a speed limit. Given the parameters $\rho_{\max}$ and $v_{\max}$, the simplest model for the relationship between v and ρ is the **constitutive equation**

$$u = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{2.23}$$

We therefore arrive at the initial value problem (IVP):

$$\left\{\begin{array}{rcll} \dfrac{\partial \rho}{\partial t} + v_{\max}\dfrac{\partial}{\partial x}\left[\rho\left(1 - \dfrac{\rho}{\rho_{\max}}\right)\right] & = & 0 & \text{for } x \in \mathbb{R},\ t > 0; \\ \rho(x,0) & = & \rho_o(x), & \text{for } x \in \mathbb{R}, \end{array}\right. \tag{2.24}$$

where we have incorporated the continuity equation in (2.19), the initial condition in (2.21), and the constitutive relation in (2.23), which is an instance of (2.22).

The partial differential equation model for traffic flow (2.24) presented in this section, based on the equation of continuity in (2.19) and a constitutive relation for the traffic velocity, u, and the traffic density ρ (of which (2.23) is just an example), was first introduced by Lighthill and Whitman in 1955 (see [LW55]); it was also treated by Richards in 1956, [Ric56]. In a subsequent section in these notes we will present an analysis of this model based on the **method of characteristics**.

We end this section with an alternate derivation of the conservation of mass equation in (2.17). In this approach we focus on the amount of fluid contained in a region B as the fluid in this region moves according to flow dictated by the velocity field $\vec{u}$. Suppose we begin to observe a portion of fluid in B at time $t = 0$. We assume that B is bounded and has smooth boundary ∂B. At some time $t > 0$, the portion of fluid in B has moved as a consequence of the fluid motion. We denote by B_t the portion of the fluid that we are following at time t (see Figure 2.1.3). To see how B_t comes about, consider a fluid element located at (x, y, z) at time $t = 0$. At time $t > 0$, the fluid element will be located at $(x(t), y(t), z(t))$, where the functions $x(t)$, $y(t)$ and $z(t)$ are solutions to the system of ordinary differential equations in (2.5) subject to the initial conditions in (2.6). We denote the point $(x(t), y(t), z(t))$ by $\varphi_t(x, y, z)$, and note that the map

$$(x, y, z) \mapsto \varphi_t(x, y, z), \qquad \text{for all } (x, y, z) \in R,$$

yields a C^1 map from R to R. Furthermore, φ_t is an invertible map for each t in the interval of existence for the initial value problem in (2.5) and (2.6). We shall refer to φ_t as the fluid flow map; it gives the location of a fluid element initially at (x, y, y) at time t as a result of fluid motion. It then follows that B_t is the image of B under the flow map φ_t; that is,

$$B_t = \varphi_t(B). \tag{2.25}$$

The total mass of the fluid in B_t is a function of time that we compute as follows

$$m(t) = \iiint_{B_t} \rho(\varphi_t(x, y, z), t)\ dV. \tag{2.26}$$

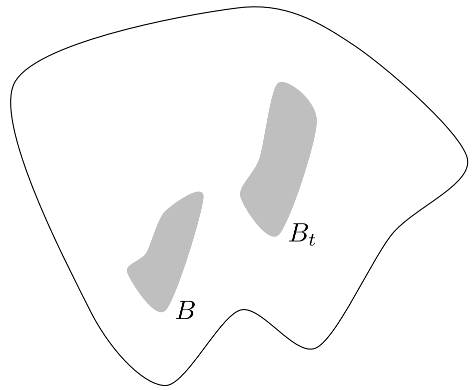

Figure 2.1.3: Balance of Forces

Note that

$$m(0) = \iiint_B \rho(x, y, y, 0)\ dV \equiv m_o, \tag{2.27}$$

which is the mass of the portion of fluid in the region B. As the flow of the fluid moves the region B, its shape might change. However, because of conservation of mass, the mass of fluid contained in B_t must the same as that contained in the region B at time $t = 0$; that is,

$$m(t) = m_o, \qquad \text{for all } t, \tag{2.28}$$

where m_o is the constant given in (2.27). It follows from (2.28) that

$$\frac{dm}{dt} = 0, \qquad \text{for all } t. \tag{2.29}$$

Before we compute $\dfrac{dm}{dt}$, we first rewrite the integral defining $m(t)$ in (2.26) by means of the change of variables provided by the flow map φ_t (see (2.25). We have

$$m(t) = \iiint_B \rho(x, y, z, t) J(x, y, z, t)\ dxdydz$$

where $J(x, y, z, t)$ the Jacobian of the map φ_t; that is, $J(x, y, z, t)$ is the determinant of the derivative map of φ_t. We then have that

$$\frac{dm}{dt} = \iiint_B \frac{\partial}{\partial t}[\rho J]\ dxdydz,$$

or

$$\frac{dm}{dt} = \iiint_B \left[\rho\frac{\partial J}{\partial t} + \frac{\partial \rho}{\partial t} J\right]\ dxdydz. \tag{2.30}$$

Making the change of variables provided by the flow map in the integral in (2.30) we obtain that

$$\begin{aligned}\frac{dm}{dt} = \iiint_{B_t} \Big[\rho(\varphi_t(x,y,z),t)\frac{1}{J(\varphi_t(x,y,z),t)}\frac{\partial}{\partial t}J(\varphi_t(x,y,z),t) \\ +\frac{\partial}{\partial t}[\rho(\varphi_t(x,y,z),t)]\Big]\ dV.\end{aligned} \tag{2.31}$$

It can be shown that

$$\frac{\partial}{\partial t}[J(\varphi_t(x,y,z),t) = J(\varphi_t(x,y,z),t)\ \nabla\cdot\vec{u}(\varphi_t(x,y,z),t), \tag{2.32}$$

see page 8 in [CM93]. Thus, substituting (2.32) into (2.31), we get

$$\frac{dm}{dt} = \iiint_{B_t}\left[(\nabla\cdot\vec{u}(\varphi_t(x,y,z),t))\rho(\varphi_t(x,y,z),t) + \frac{\partial}{\partial t}[\rho(\varphi_t(x,y,z),t)]\right]\ dV,$$

which we can write as

$$\frac{dm}{dt} = \iiint_{B_t}\left[(\nabla\cdot\vec{u})\rho + \frac{D\rho}{Dt}\right]\ dV, \tag{2.33}$$

where we have set

$$\begin{aligned}\frac{D\rho}{Dt} &= \frac{\partial}{\partial t}[\rho(\varphi_t(x,y,z),t)] \\ &= \frac{\partial}{\partial t}[\rho(x(t),y(t),z(t),t)] \\ &= \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt} + \frac{\partial\rho}{\partial t},\end{aligned} \tag{2.34}$$

where we have used the Chain Rule in the last step of the calculations in (2.34) and assumed that the density ρ is a C^1 field. We therefore have that

$$\frac{\partial}{\partial t}[\rho(x(t),y(t),z(t),t)] = \frac{\partial\rho}{\partial t} + u_1\frac{\partial\rho}{\partial x} + u_2\frac{\partial\rho}{\partial y} + u_3\frac{\partial\rho}{\partial z}, \tag{2.35}$$

where we have used the fact that $(x(t),y(t),z(t))$ solves the system of ordinary differential equations in (2.5). Writing (2.35) in vector notation we obtain

$$\frac{\partial}{\partial t}[\rho(x(t),y(t),z(t),t)] = \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho, \tag{2.36}$$

where $\nabla\rho = \left(\frac{\partial\rho}{\partial x},\frac{\partial\rho}{\partial y},\frac{\partial\rho}{\partial z}\right)$ is the gradient of ρ. The expression in (2.36) is called the **material derivative** of the field ρ. It is also referred to as the **convective derivative** of ρ and is usually denoted by $\frac{D\rho}{Dt}$, so that

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho. \tag{2.37}$$

In general, given a C^1 scalar field, g, defined in a region R, the material derivative of g is given by

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \vec{u} \cdot \nabla g. \tag{2.38}$$

The material derivative of g in (2.38) expresses the rate of change of g along the pathlines as a result of the fact that the field g might change in time as well as a result of the motion of the fluid. The material derivative of a C^1 vector field, $\overrightarrow{G} = (g_1, g_2, g_3)$, is

$$\frac{D\overrightarrow{G}}{Dt} = \left(\frac{Dg_1}{Dt}, \frac{Dg_2}{Dt}, \frac{Dg_3}{Dt} \right),$$

which can be written as

$$\frac{D\overrightarrow{G}}{Dt} = \frac{\partial \overrightarrow{G}}{\partial t} + (\vec{u} \cdot \nabla)\overrightarrow{G}. \tag{2.39}$$

Combining (2.29) with (2.33) we get that

$$\iiint_{B_t} \left[(\nabla \cdot \vec{u})\rho + \frac{D\rho}{Dt} \right] dV = 0, \quad \text{for all } t. \tag{2.40}$$

It follows from (2.40) that

$$\frac{D\rho}{Dt} + (\nabla \cdot \vec{u})\rho = 0, \quad \text{in } R, \text{ for all } t, \tag{2.41}$$

where the material derivative, $\frac{D\rho}{Dt}$, of ρ is given in (2.37); that is,

$$\begin{aligned} \frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) - (\nabla \cdot \vec{u})\rho \end{aligned} \tag{2.42}$$

substituting the result of the calculations in (2.42) into (2.41) then yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad \text{in } R, \text{ for all } t,$$

which is the continuity equation in (2.17). We have also shown that the equation in (2.41) is an equivalent form of the continuity equation. We shall rewrite it here as

$$\frac{D\rho}{Dt} = -(\nabla \cdot \vec{u})\rho, \quad \text{in } R, \text{ for all } t. \tag{2.43}$$

2.1.2 Conservation of Momentum for an Ideal Fluid

The total momentum at time t of a the portion of fluid contained in a region B_t with smooth boundary, ∂B_t, is given by

$$\overrightarrow{\Pi}_B(t) = \iiint_{B_t} \rho(x(t), y(t), z(t), t)\vec{u}(x(t), y(t), z(t), t)\ dV,$$

or

$$\overrightarrow{\Pi}_B(t) = \iiint_{B_t} \rho(\varphi_t(x, y, x), t)\vec{u}(\varphi_t(x, y, x), t)\ dV,$$

and which we'll simply write as

$$\overrightarrow{\Pi}_B(t) = \iiint_{B_t} \rho\vec{u}\ dV, \tag{2.44}$$

(see Figure 2.1.3). The principle of conservation of momentum states that the rate of change of the total momentum of the fluid in B_t has to be accounted for by the balance of forces acting on B_t:

$$\frac{d\overrightarrow{\Pi}_B}{dt} = \text{ Balance of Forces on } B_t; \tag{2.45}$$

this is, in fact, Newton's second law of motion.

There are two types of forces acting on the portion of fluid in B_t that contribute to the balance of forces in the right–hand side of the equation in (2.45). There are forces of stress due to the fluid surrounding the region B_t, and there are external, or body forces, such as gravity or electromagnetic forces. We can then rewrite the conservation of momentum equation in (2.45) as

$$\frac{d\overrightarrow{\Pi}_B}{dt} = \overrightarrow{S}_B(t) + \overrightarrow{F}_B(t), \tag{2.46}$$

where $\overrightarrow{S}_B(t)$ denotes the total vector sum of the stress forces acting on B_t, and $\overrightarrow{F}_B(t)$ the total vector sum of body forces acting on B_t.

We assume that

$$\overrightarrow{F}_B(t) = \iiint_{B_t} \vec{f}\ dV, \tag{2.47}$$

where the vector field $\vec{f}(x, y, z, t)$ gives the total forces per unit volume acting on an element of fluid around the point (x, y, z) at time t.

In this section we shall make a special assumption when modeling the stress forces acting on the fluid. We assume that the fluid under consideration is an **ideal fluid**. This means that at any point, (x, y, y), on a surface in the fluid, the stress force per unit area exerted across the surface is given by

$$p(x, y, z, t)\vec{n}$$

where $\vec{n}$ is a unit vector perpendicular to the surface at (x, y, z) and time t, and $p(x, y, z, t)$ is a scalar field called the **pressure**. It then follows that

$$\overrightarrow{S}_B(t) = -\iint_{\partial B_t} p\vec{n}\; dA, \tag{2.48}$$

where $\vec{n}$ is the outward unit normal to ∂B_t.

Substituting the expressions in (2.48) and (2.47) into the conservation of momentum expression in (2.46) yields

$$\frac{d\overrightarrow{\Pi}_B}{dt} = -\iint_{\partial B_t} p\vec{n}\; dA + \iiint_{B_t} \vec{f}\; dV. \tag{2.49}$$

Writing the unit vector $\vec{n}$ in Cartesian coordinates, (n_1, n_2, n_3), we see that the stress forces term in (2.49) has components

$$-\iint_{\partial B_t} pn_1\; dA, \; -\iint_{\partial B_t} pn_2\; dA, \text{ and } -\iint_{\partial B_t} pn_3\; dA.$$

Applying the divergence theorem to each of these components we get

$$-\iint_{\partial B_t} pn_1\; dA = -\iiint_{B_t} \frac{\partial p}{\partial x}\; dV$$

$$-\iint_{\partial B_t} pn_2\; dA = -\iiint_{B_t} \frac{\partial p}{\partial y}\; dV$$

and

$$-\iint_{\partial B_t} pn_3\; dA = -\iiint_{B_t} \frac{\partial p}{\partial z}\; dV.$$

Substituting these expressions into the definition of $\overrightarrow{S}_B(t)$ in (2.48) we obtain

$$\vec{S}_B(t) = -\iiint_{B_t} \nabla p\; dV, \tag{2.50}$$

where

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) \tag{2.51}$$

is the gradient of p. Combining (2.50), (2.48) and (2.46), we can rewrite the conservation of momentum equation in (2.49) as

$$\frac{d\overrightarrow{\Pi}_B}{dt} = -\iiint_{B_t} \nabla p\; dV + \iiint_{B_t} \vec{f}\; dV, \tag{2.52}$$

where ∇p is as given in (2.51).

Next, we see how to compute the left–hand side of the equation in (2.52),

$$\frac{d\overrightarrow{\Pi}_B}{dt} = \frac{d}{dt}\iiint_{B_t} \rho\vec{u}\; dV, \tag{2.53}$$

according to the definition of momentum in (2.44).

Observe that, since B_t comes about as the result of the action of the flow map φ_t on B (see 2.25), we can rewrite the integral on the right–hand side of (2.53) as

$$\begin{aligned}\iiint_{B_t} \rho\vec{u}\ dV &= \iiint_{B_t} \rho(\varphi_t(x,y,z),t)\vec{u}(\varphi_t(x,y,z),t)\ dV \\ &= \iiint_{B} \rho(x,y,z,t)\vec{u}(x,y,z,t)J(x,y,z,t)\ dxdydz\end{aligned}$$

where $J(x,y,z,t)$ the Jacobian of the map φ_t; that is, $J(x,y,z,t)$ is the determinant of the derivative map of φ_t. We then have that

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B} \frac{\partial}{\partial t}[J\rho\vec{u}]\ dxdydz,$$

or

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B} \left[\frac{\partial J}{\partial t}\rho\vec{u} + \frac{\partial}{\partial t}[\rho\vec{u}]J\right]\ dxdydz. \tag{2.54}$$

Substituting (2.32) into (2.54) yields

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B} \left[(\nabla\cdot\vec{u})\rho\vec{u} + \frac{\partial}{\partial t}[\rho\vec{u}]\right] J\ dxdydz,$$

which can be written as

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B_t} \left[(\nabla\cdot\vec{u})\rho\vec{u} + \frac{\partial}{\partial t}[\rho\vec{u}]\right]\ dV. \tag{2.55}$$

Using the expression for the material derivative of a vector field in (2.39), we can rewrite (2.55) as

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B_t} \left[\frac{D}{Dt}(\rho\vec{u}) + (\nabla\cdot\vec{u})\rho\vec{u}\right]\ dV. \tag{2.56}$$

Using the definition of the convective derivative for a vector field in (2.39) we have that

$$\frac{D}{Dt}(\rho\vec{u}) = \rho\frac{D\vec{u}}{Dt} + \frac{D\rho}{Dt}\vec{u}, \tag{2.57}$$

where

$$\begin{aligned}\frac{D\rho}{Dt} &= \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho \\ &= \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) - \rho\nabla\cdot\vec{u};\end{aligned}$$

it then follows from the conservation mass equation in (2.17) that

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{u}, \tag{2.58}$$

which is an alternate form of the conservation of mass principle.

Combining (2.57) and (2.58) then yields

$$\frac{D}{Dt}(\rho\vec{u}) = \rho\frac{D\vec{u}}{Dt} - (\nabla\cdot\vec{u})\rho\vec{u}. \tag{2.59}$$

Substituting the expression for $\frac{D}{Dt}(\rho\vec{u})$ in (2.59) into the expression for the rate of change of momentum in (2.56) yields

$$\frac{d}{dt}\iiint_{B_t}\rho\vec{u}\;dV = \iiint_{B_t}\rho\frac{D\vec{u}}{Dt}\;dV. \tag{2.60}$$

Substituting the expression for the rate of change of momentum in (2.60) into the left–hand side of (2.52) yields

$$\iiint_{B_t}\rho\frac{D\vec{u}}{Dt}\;dV = -\iiint_{B_t}\nabla p\;dV + \iiint_{B_t}\vec{f}\;dV,$$

or

$$\iiint_{B_t}\left[\rho\frac{D\vec{u}}{Dt} + \nabla p - \vec{f}\right]dV = 0, \quad \text{for all } t. \tag{2.61}$$

Assuming that the fields ρ, $\vec{u}$ and p are C^1 over R and for all times t, and that the field $\vec{f}$ is continuous over R and for all times t, we see that the integrand in the left–hand side of (2.61) is continuous over R and for all times t. Thus, since (2.61) holds true for all bounded subregions, B_t, of R with smooth boundary, we conclude that

$$\rho\frac{D\vec{u}}{Dt} + \nabla p - \vec{f} = 0, \quad \text{in } R, \text{ for all } t,$$

or

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p + \vec{f}, \quad \text{in } R, \text{ for all } t, \tag{2.62}$$

which is the differential form of the conservation of momentum principle.

Observe that the PDE in (2.62) is a vector differential equation in three dimensions. As such, it is really a system of three first–order PDEs:

$$\begin{cases} \rho\dfrac{Du_1}{Dt} &= -\dfrac{\partial p}{\partial x} + f_1; \\[2ex] \rho\dfrac{Du_2}{Dt} &= -\dfrac{\partial p}{\partial y} + f_2; \\[2ex] \rho\dfrac{Du_3}{Dt} &= -\dfrac{\partial p}{\partial z} + f_3. \end{cases} \tag{2.63}$$

The equations in (2.18) and (2.63) constitute a system of four first–order PDEs in the (possibly) unknown scalar fields u_1, u_2, u_3, ρ and p (the body forces field $\vec{f}$ can usually be determined from the outset). Thus, in order to have any hope for solving the system of conservation equations in (2.17) and (2.62), we need to have at least one more relation, or equation, involving the velocity field, $\vec{u}$, the density, ρ, and the pressure, p. Another relation will be provided by the principle of conservation of energy to be discussed in the next section.

The expression in (2.60) holds true for any C^1 vector field $\overrightarrow{G}$ in R,

$$\frac{d}{dt}\iiint_{B_t} \rho\overrightarrow{G}\ dV = \iiint_{B_t} \rho\frac{D\overrightarrow{G}}{Dt}\ dV,$$

or any C^1 scalar field g,

$$\frac{d}{dt}\iiint_{B_t} \rho g\ dV = \iiint_{B_t} \rho\frac{Dg}{Dt}\ dV, \tag{2.64}$$

where $\dfrac{Dg}{Dt}$ is the material derivative of g. This is known as the **Transport Theorem**. We will have opportunity to apply the transport theorem in (2.64) in the next section.

2.1.3 Conservation of Energy in Incompressible Flow

Consider the volume of the portion of the fluid in B_t at time t (see Figure 2.1.3),

$$v(t) = \iiint_{B_t} dV. \tag{2.65}$$

As the shape of the region B_t changes with the flow, the volume of B_t might change also. We compute the rate at which the volume changes by first rewriting the expression for $v(t)$ in (2.65) as

$$v(t) = \iiint_{B} J(\varphi_t(x,y,z),t)\ dV. \tag{2.66}$$

It follows from (2.66) that

$$\frac{dv}{dt} = \iiint_{B} \frac{\partial}{\partial t}[J(\varphi_t(x,y,z),t)]\ xdydz, \tag{2.67}$$

where

$$\frac{\partial}{\partial t}[J(\varphi_t(x,y,z),t) = (\nabla\cdot\vec{u}(\varphi_t(x,y,z),t))J(\varphi_t(x,y,z),t)\ ,$$

according to (2.32). We therefore obtain from (2.67) that

$$\frac{dv}{dt} = \iiint_{B}(\nabla\cdot\vec{u}(\varphi_t(x,y,z),t))J(\varphi_t(x,y,z),t)\ dxdydz,$$

which we can rewrite as

$$\frac{dv}{dt} = \iiint_{B_t} \nabla\cdot\vec{u}\ dV. \tag{2.68}$$

In an **incompressible flow** the volume of any portion of the fluid does not change with time. We therefore obtain from (2.68) that

$$\iiint_{B_t} \nabla \cdot \vec{u}\; dV = 0, \qquad \text{for all } t. \tag{2.69}$$

Since the expression in (2.69) holds true for any B_t in R, it follows that, for case in which the velocity field, $\vec{u}$, is C^1 in R, the condition for the flow to be incompressible is

$$\nabla \cdot \vec{u} = 0, \qquad \text{in } R \text{ for all } t. \tag{2.70}$$

We show in this section that, in an ideal incompressible fluid, the kinetic energy in the portion of the fluid in B_t is conserved.

The kinetic energy of the portion of the fluid B_t at time t is given by

$$E(t) = \frac{1}{2}\iiint_{B_t} \rho\|\vec{u}\|^2\; dV, \tag{2.71}$$

where $\|u\|^2 = \vec{u}\cdot\vec{u}$ is the square of the Euclidean norm of the velocity field $\vec{u}$.

The rate of change of E in (2.71) is given by the Transport Theorem in (2.64) to be

$$\frac{dE}{dt} = \frac{1}{2}\iiint_{B_t} \rho\frac{D}{Dt}[\|\vec{u}\|^2]\; dV, \tag{2.72}$$

where

$$\frac{D}{Dt}[\|\vec{u}\|^2] = 2\vec{u}\cdot\frac{D\vec{u}}{Dt},$$

so that, in view of (2.72),

$$\frac{dE}{dt} = \iiint_{B_t} \rho\vec{u}\cdot\frac{D\vec{u}}{Dt}\; dV,$$

or

$$\frac{dE}{dt} = \iiint_{B_t} \vec{u}\cdot\left(\rho\frac{D\vec{u}}{Dt}\right)\; dV. \tag{2.73}$$

Substituting the law of conservation of momentum expression for an ideal fluid in (2.62) into the right–hand side of (2.73) then yields

$$\frac{dE}{dt} = \iiint_{B_t} \vec{u}\cdot\left(-\nabla p + \vec{f}\right)\; dV,$$

which can be written as

$$\frac{dE}{dt} = -\iiint_{B_t} \nabla p\cdot\vec{u}\; dV + \iiint_{B_t} \vec{f}\cdot\vec{u}\; dV. \tag{2.74}$$

The right–most integral in (2.74) measures the rate at which body forces do work in the portion of fluid in B_t at time t. In order to understate the other integral in (2.74) we use the assumption that the fluid is incompressible, stated as the PDE in (2.70), to obtain

$$\nabla \cdot (p\vec{u}) = \nabla p \cdot \vec{u} + p\nabla \cdot \vec{u} = \nabla p \cdot \vec{u},$$

so that

$$\iiint_{B_t} \nabla p \cdot \vec{u}\; dV = \iiint_{B_t} \nabla \cdot (p\vec{u})\; dV. \tag{2.75}$$

Applying the divergence theorem to the integral on the right–hand side of (2.75) yields

$$\iiint_{B_t} \nabla p \cdot \vec{u}\; dV = \iint_{\partial B_t} p\vec{u} \cdot \vec{n}\; dA, \tag{2.76}$$

where $\vec{n}$ denotes the outward unit normal vector the boundary of B_t. Substituting the expression in (2.76) into the right–hand side of (2.74) then yields

$$\frac{dE}{dt} = -\iint_{\partial B_t} p\vec{u} \cdot \vec{n}\; dA + \iiint_{B_t} \vec{f} \cdot \vec{u}\; dV. \tag{2.77}$$

Observe that $-\displaystyle\iint_{\partial B_t} p\vec{u} \cdot \vec{n}\; dA$ gives the rate at which the stress forces are doing work on the portion of fluid in B_t. Hence, the equation in (2.77) is a statement of the conservation of kinetic energy.

2.1.4 Euler Equations for Incompressible, Ideal Fluids

Putting together the PDEs in (2.58), (2.62) and (2.70) we obtain the system of PDEs

$$\begin{cases} \dfrac{D\rho}{Dt} &= 0; \\[2ex] \rho\dfrac{D\vec{u}}{Dt} &= -\nabla p + \vec{f}; \\[2ex] \nabla \cdot \vec{u} &= 0, \end{cases} \tag{2.78}$$

stating the principles of conservation of mass, conservation of momentum, and conservation of energy, respectively, for incompressible, ideal fluids. The equations in the system of PDEs in (2.78) are known as the Euler equations for incompressible, ideal fluids. Using the definition of the material derivative, $\dfrac{D}{Dt}$, in (2.38) and (2.39), the Euler equations in (2.78) can also be written as

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \vec{u} \cdot \nabla\rho &= 0; \\[2ex] \rho\dfrac{\partial\vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} &= -\nabla p + \vec{f}; \\[2ex] \nabla \cdot \vec{u} &= 0, \end{cases} \tag{2.79}$$

The fields ρ, $\vec{u}$ and p in (2.79) are assumed to be C^1 functions defined in and open region R in $\mathbb{R}^3$, and for $t \geqslant 0$; the field $\vec{f}$ is assumed to be continuous in R and for all $t \geqslant 0$. The field $\vec{f}$ is usually know; but the functions ρ, $\vec{u}$ and p are unknown. We would like to obtain information about these functions for all times, t, and all points in R, given some initial conditions; for example,

$$\begin{cases} \rho(x,y,z,0) = \rho_o(x,y,z), & \text{for } (x,y,z) \in R; \\ \vec{u}(x,y,z,0) = \vec{u}_o(x,y,z), & \text{for } (x,y,z) \in R; \\ p(x,y,z,0) = p_o(x,y,z),, & \text{for } (x,y,z) \in R, \end{cases}$$

where ρ_o, $\vec{u}_o$ and p_o are given functions defined in R. Since, we want the flow to remain within the region R, we also impose the **boundary condition**

$$\vec{u} \cdot \vec{n} = 0, \quad \text{on } \partial R, \text{ for all } t, \tag{2.80}$$

where we are assuming that R has a smooth boundary ∂R. The condition in (2.80) forbids fluid to cross in or out of the boundary.

2.2 Modeling Diffusion

The random migration of small particles (e.g., pollen grains, large molecules, etc.) immersed in a stationary fluid is known as **diffusion**. This process, also known as Brownian motion, is caused by the random bombardment of the particles by the fluid molecules because of thermal excitation. Brownian motion can be modeled probabilistically by looking at motions of large ensemble of particles. This is a microscopic view. In this section we would like to provide a macroscopic model of diffusion based on a conservation principle.

Imagine that a certain number of Brownian particles moves within a region R in $\mathbb{R}^3$ pictured in Figure 2.2.4. Assume that there is a vector field $\overrightarrow{J}$ that

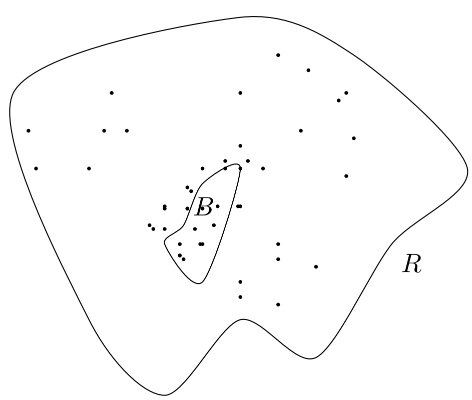

Figure 2.2.4: Brownian Particles in a Region R

gives a measure of the number of particles that cross a unit cross–sectional at point $(x, y, z) \in R$ and time t as follows

$$\overrightarrow{J}(x, y, z, t) \cdot \vec{n} dA$$

gives, approximately, the number of particles that cross a small section of the surface of area dA, per unit time, in a direction perpendicular to the surface at that point. It then follows that the number of particles per unit time crossing the smooth boundary of a region $B \subset R$ into that region (see Figure 2.2.4) is given by

$$-\iint_{\partial B} \overrightarrow{J}(x, y, z, t) \cdot \vec{n}\ dA, \tag{2.81}$$

where the minus sign in (2.81) takes into account that we are taking $\vec{n}$ to be the outward unit normal to ∂B. The expression in (2.81) is called the **flux** of particles across the boundary of B.

Assume that the concentration of particles in the region R at any time t is given by a C^1 scalar field, u, so that number of particles contained in the region B is given at time t is given by

$$N_B(t) = \iiint_B u(x, y, x, t)\ dxdydx, \qquad \text{for all } t. \tag{2.82}$$

Assuming that particles are not being created or destroyed, we get the conservation principle

$$\frac{dN_B}{dt} = -\iint_{\partial B} \overrightarrow{J}(x, y, z, t) \cdot \vec{n}\ dA \tag{2.83}$$

Since we are assuming that u is a C^1 field, we can differentiate under the integral sign in (2.82) to rewrite (2.83) as

$$\iiint_B \frac{\partial u}{\partial t}\ dxdydx = -\iint_{\partial B} \overrightarrow{J}(x, y, z, t) \cdot \vec{n}\ dA \tag{2.84}$$

If we also assume that the vector field $\overrightarrow{J}$ is a C^1 function, we can use the Divergence Theorem to rewrite the right–hand side of (2.84) to obtain

$$\iiint_B \frac{\partial u}{\partial t}\ dV = -\iiint_B \nabla \cdot \overrightarrow{J}\ dV$$

or

$$\iiint_B \left[\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{J}\right]\ dV = 0. \tag{2.85}$$

Since (2.85) holds true for all bounded subsets, B, of R, and all times t, we obtain the PDE

$$\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{J} = 0, \quad \text{in } R, \text{ for all } t. \tag{2.86}$$

The PDE in (2.86) has two unknown functions: the concentration u and the flux field $\overrightarrow{J}$. Thus, in order to complete the modeling, we need a constitutive equation relating u and $\overrightarrow{J}$. This is provided by Fick's First Law of Diffusion (see [Ber83, pg. 18]):

$$\overrightarrow{J} = -D\nabla \cdot u, \quad \text{in } R, \text{ for all } t, \tag{2.87}$$

where D is a proportionality constant known as the diffusion constant of the medium in which the particles are, or diffusivity. Observe that D in (2.87) has units of squared length per time. The expression in (2.87) postulates that the flux of Brownian particles is proportional to the negative gradient of the concentration. Thus, the diffusing particles will move from regions of high concentration to regions of low concentration.

Substituting the expression for $\overrightarrow{J}$ in (2.87) into the conservation equation in (2.86) we obtain

$$\frac{\partial u}{\partial t} - D\nabla \cdot (\nabla u) = 0, \quad \text{in } R, \text{ for all } t, \tag{2.88}$$

where we have used the assumption that D is constant.

Assuming that u is also a C^2 function, we can use the definitions of gradient and divergence to compute

$$\nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.89}$$

The expression on the right–hand side of (2.89) is known as the **Laplacian** of u, and is usually denoted by the symbol Δu, so that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.90}$$

Another notation for Δu found in various textbooks is $\nabla^2 u$.

In view of (2.89) and (2.90), we see that the PDE in (2.88) can be written as

$$\frac{\partial u}{\partial t} = D\Delta u, \quad \text{in } R, \text{ for all } t, \tag{2.91}$$

which is called the **diffusion equation**. The expression in (2.91) is also known as Fick's second equation (see [Ber83, pg. 20]), or Fick's Second Law of Diffusion.

For the case of in which the diffusing substance is constrained to move in one space direction (say, parallel to the x–axis), the diffusion equation in (2.91) becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{2.92}$$

The equation in (2.92) applies to the situation in which medium containing Brownian particles is in a cylindrical region of constant cross sectional area and axis parallel to the x–axis. In later chapter in these notes, we will show how to solve the PDE in (2.92) over the entire real line subject to an initial condition

$$u(x, 0) = f(x), \qquad \text{for all } x \in \mathbb{R},$$

for some given function $f\colon \mathbb{R} \to \mathbb{R}$, and some integrability conditions on u, $\dfrac{\partial u}{\partial x}$ and f.

The equation in (2.92) also describes the flow of heat in a cylindrical metal rod of constant cross–sectional area whose cylindrical boundary is insulated so that heat can only flow in or out of the rod through the cross sections a the ends of the rod (see Assignment #4). In this case $u(x,t)$ denotes the temperature in

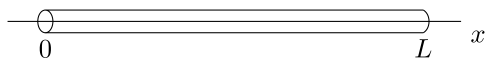

Figure 2.2.5: Heat Conduction in a Cylindrical Rod

the cross–section of the rod located at x and at time t, and the constant D is given by

$$D = \frac{\kappa}{c\rho},$$

where ρ is the density, c is the specific heat, and κ is the heat conductivity of the material the of the rod (see Assignment #4). Thus, (2.92) ia also called the **heat equation**. In this case D is called the **thermal diffusivity**.

In these notes we will see how to solve the heat equation in (2.92) subject the initial and boundary conditions

$$\begin{cases} u(x,0) = f(x), & \text{for } 0 < x < L; \\ u(0,t) = T_o(t), & \text{for } t > 0; \\ u(L,t) = T_L(t), & \text{for } t > 0, \end{cases}$$

where f, T_o and T_L are given functions of single variable. We will also solve the problem with the boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial x}(0,t) = 0, & \text{for } t > 0; \\ \dfrac{\partial u}{\partial x}(L,t) = 0, & \text{for } t > 0. \end{cases}$$

These conditions imply that heat cannot flow through the end cross–sections either; so that the rod is totally insulated.

2.3 Variational Problems

In the previous two sections we have seen how conservation principles give rise to problems involving PDEs. Another important source of PDE problems arises from the application of **variational principles**. A variational principle states that a configuration, or function, describing the state of a system must minimize, or maximize, certain quantity (e.g., energy). In this section we will see two applications of variational principles: the derivations of the minimal surface equation and the vibrating string equation.

2.3.1 Minimal Surfaces

Imagine you take a twisted wire loop, as that pictured in Figure 2.3.6, and dip into a soap solution. When you pull it out of the solution, a soap film spanning the wire loop develops. We are interested in understanding the mathematical properties of the film, which can be modeled by a smooth surface in three

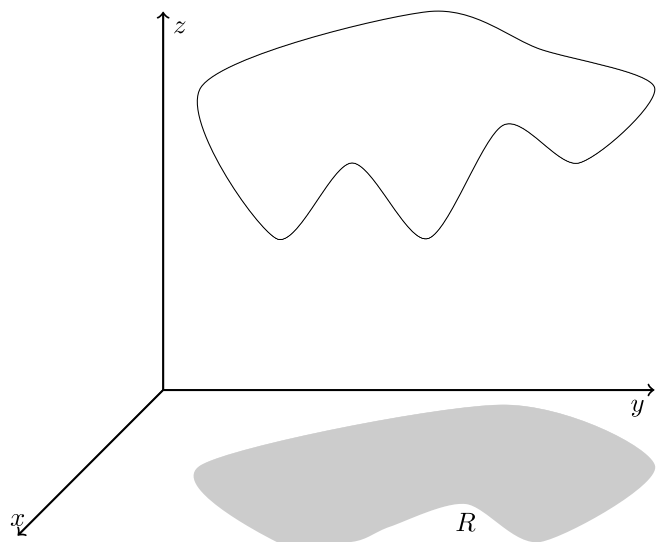

Figure 2.3.6: Wire Loop

dimensional space. Specifically, the shape of the soap film spanning the wire loop, can be modeled by the graph of a smooth function, $u\colon \overline{R} \to \mathbb{R}$, defined on the closure of a bounded region, R, in the xy–plane with smooth boundary ∂R. The physical explanation for the shape of the soap film relies on the variational principle that states that, at equilibrium, the configuration of the film must be such that the energy associated with the surface tension in the film must be the lowest possible. Since the energy associated with surface tension in the film is proportional to the area of the surface, it follows from the least–energy principle that a soap film must minimize the area; in other words, the soap film spanning the wire loop must have the shape of a smooth surface in space containing the wire loop with the property that it has the smallest possible area among all smooth surfaces that span the wire loop. In this section we will develop a mathematical formulation of this variational problem.

The wire loop can be modeled by the curve determined by the set of points:

$$(x, y, g(x, y)), \qquad \text{for } (x, y) \in \partial R,$$

where ∂R is the smooth boundary of a bounded open region R in the xy–plane (see Figure 2.3.6), and g is a given function defined in a neighborhood of ∂R, which is assumed to be continuous. A surface, S, spanning the wire loop can be modeled by the image of a C^1 map

$$\Phi\colon R \to \mathbb{R}^3$$

given by

$$\Phi(x, y) = (x, y, u(x, u)), \qquad \text{for all } x \in \overline{R}, \tag{2.93}$$

where $\overline{R} = R \cup \partial R$ is the closure of R, and

$$u\colon \overline{R} \to \mathbb{R}$$

is a function that is assumed to be C^2 in R and continuous on $\overline{R}$; we write

$$u \in C^2(R) \cap C(\overline{R}).$$

Let $\mathcal{A}_g$ denote the collection of functions $u \in C^2(R) \cap C(\overline{R})$ satisfying

$$u(x, y) = g(x, y), \quad \text{for all } (x, y) \in \partial R;$$

that is,

$$\mathcal{A}_g = \{u \in C^2(R) \cap C(\overline{R}) \mid u = g \text{ on } \partial R\}. \tag{2.94}$$

Next, we see how to compute the area of the surface $S_u = \Phi(R)$, where Φ is the map given in (2.93) for $u \in \mathcal{A}_g$, where $\mathcal{A}_g$ is the class of functions defined in (2.94).

The grid lines $x = c$ and $y = d$, for arbitrary constants c and d, are mapped by the parametrization Φ into curves in the surface S_u given by

$$y \mapsto \Phi(c, y)$$

and

$$x \mapsto \Phi(x, d),$$

respectively. The tangent vectors to these paths are given by

$$\Phi_y = \left(0, 1, \frac{\partial u}{\partial y}\right) \tag{2.95}$$

and

$$\Phi_x = \left(1, 0, \frac{\partial u}{\partial x}\right), \tag{2.96}$$

respectively. The quantity

$$\|\Phi_x \times \Phi_y\| \Delta x \Delta y \tag{2.97}$$

gives an approximation to the area of portion of the surface S_u that results from mapping the rectangle $[x, x + \Delta x] \times [y, y + \Delta y]$ in the region R to the surface S_u by means of the parametrization Φ given in (2.93). Adding up all the contributions in (2.97), while refining the grid, yields the following formula for the area S_u:

$$\text{area}(S_u) = \iint_R \|\Phi_x \times \Phi_y\| \, dxdy. \tag{2.98}$$

Using the definitions of the tangent vectors Φ_x and Φ_y in (2.95) and (2.96), respectively, we obtain that

$$\Phi_x \times \Phi_y = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right),$$

so that

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2},$$

or

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + |\nabla u|^2},$$

where $|\nabla u|$ denotes the Euclidean norm of ∇u. We can therefore write (2.98) as

$$\text{area}(S_u) = \iint_R \sqrt{1 + |\nabla u|^2}\; dxdy. \tag{2.99}$$

The formula in (2.99) allows us to define a map

$$A\colon \mathcal{A}_g \to \mathbb{R}$$

by

$$A(u) = \iint_R \sqrt{1 + |\nabla u|^2}\; dxdy, \quad \text{for all } u \in \mathcal{A}_g, \tag{2.100}$$

which gives the area of the surface parametrized by the map $\Phi\colon \overline{R} \to \mathbb{R}^3$ given in (2.93) for $u \in \mathcal{A}_g$. We will refer to the map $A\colon \mathcal{A}_g \to \mathbb{R}$ defined in (2.100) as the area **functional**. With the new notation we can restate the variational problem of this section as follows:

Problem 2.3.1 (Variational Problem 1)**.** *Out of all functions in $\mathcal{A}_g$, find one such that*

$$A(u) \leqslant A(v), \quad \text{for all } v \in \mathcal{A}_g. \tag{2.101}$$

That is, find a function in $\mathcal{A}_g$ that minimizes the area functional in the class $\mathcal{A}_g$.

Problem 2.3.1 is an instance of what has been known as Plateau's problem in the Calculus of Variations. The mathematical question surrounding Pateau's problem was first formulated by Euler and Lagrange around 1760. In the middle of the 19^{th} century, the Belgian physicist Joseph Plateu conducted experiments with soap films that led him to the conjecture that soap films that form around wire loops are of minimal surface area. It was not until 1931 that the American mathematician Jesse Douglas and the Hungarian mathematician Tibor Radó, independently, came up with the first mathematical proofs for the existence of minimal surfaces. In this section we will derive a necessary condition for the existence of a solution to Problem 2.3.1, which is expressed in terms of a PDE that $u \in \mathcal{A}_g$ must satisfy, the minimal surface equation.

Suppose we have found a solution, $u \in \mathcal{A}_g$, to Problem 2.3.1 in $u \in \mathcal{A}_g$. Let $\varphi\colon \overline{R} \to \mathbb{R}$ by a C^∞ function with compact support in R; we write $\varphi \in C_c^\infty(R)$ (see Assignment #5 for a construction of such function). It then follows that

$$u + t\varphi \in \mathcal{A}_g, \quad \text{for all } t \in \mathbb{R}, \tag{2.102}$$

since φ vanishes in a neighborhood of ∂R and therefore $u + t\varphi = g$ on ∂R. It follows from (2.102) and (2.101) that

$$A(u) \leqslant A(u + t\varphi), \quad \text{for all } t \in \mathbb{R}. \tag{2.103}$$

Consequently, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = A(u + t\varphi), \quad \text{for all } t \in \mathbb{R}, \tag{2.104}$$

has a minimum at 0, by virtue of (2.104) and (2.104). It follows from this observation that, if f is differentiable at 0, then

$$f'(0) = 0. \tag{2.105}$$

We will see next that, since we are assuming that $u \in C^2(R) \cap C(\overline{R})$ and $\varphi \in C_c^\infty(R)$, f is indeed differentiable. To see why this is the case, use (2.104) and (2.100) to compute

$$f(t) = \iint_R \sqrt{1 + |\nabla(u + t\varphi)|^2}\; dxdy, \quad \text{for all } t \in \mathbb{R}, \tag{2.106}$$

where

$$\nabla(u + t\varphi) = \nabla u + t\nabla\varphi, \quad \text{for all } t \in \mathbb{R},$$

by the linearity of the differential operator ∇. It then follows that

$$\begin{aligned} \nabla(u + t\varphi) &= (\nabla u + t\nabla\varphi) \cdot (\nabla u + t\nabla\varphi) \\ &= \nabla u \cdot \nabla u + t\nabla u \cdot \nabla\varphi + t\nabla\varphi \cdot \nabla u + t^2 \nabla\varphi \cdot \nabla\varphi \\ &= |\nabla u|^2 + 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2, \end{aligned}$$

so that, substituting into (2.106),

$$f(t) = \iint_R \sqrt{1 + |\nabla u|^2 + 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2}\; dxdy, \quad \text{for all } t \in \mathbb{R}. \tag{2.107}$$

Since the integrand in (2.107) is C^1, we can differentiate under the integral sign to get

$$f'(t) = \iint_R \frac{\nabla u \cdot \nabla\varphi + t|\nabla\varphi|^2}{\sqrt{1 + |\nabla u|^2 + 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2}}\; dxdy, \tag{2.108}$$

for all $t \in \mathbb{R}$. Thus, f is differentiable and, substituting 0 for t in (2.108),

$$f'(0) = \iint_R \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}}\, dxdy. \tag{2.109}$$

Hence, if u is a minimizer of the area functional in $\mathcal{A}_g$, it follows from (2.104) and (2.109) that

$$\iint_R \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}}\, dxdy = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{2.110}$$

The statement in (2.110) provides a necessary condition for the existence of a minimizer of the area functional in $\mathcal{A}_g$. We will next see how (2.110) gives rise to a PDE that $u \in C^2(R) \cap C(\overline{R})$ must satisfy in order for it to be minimizer of the area functional in $\mathcal{A}_g$.

First, we "integrate by parts" (see Assignment #6) in (2.110) to get

$$-\iint_R \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \varphi\, dxdy + \int_{\partial R} \varphi \frac{\nabla u \cdot \vec{n}}{\sqrt{1 + |\nabla u|^2}}\, ds = 0, \tag{2.111}$$

for all $\varphi \in C_c^\infty(R)$, where the second integral in (2.111) is a path integral around the boundary of R. Since $\varphi \in C_c^\infty(R)$ vanishes in a neighborhood of the boundary of R, it follows from (2.111) that

$$\iint_R \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \varphi\, dxdy = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{2.112}$$

By virtue of the assumption that u is a C^2 functions, it follows that the divergence term of the integrand (2.112) is continuous on R, it follows from the statement in (2.112) that

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0, \quad \text{in } R. \tag{2.113}$$

(see Assignment #6).

The equation in (2.113) is a second order nonlinear PDE known as the **minimal surface equation**. It provides a necessary condition for a function $u \in C^2(R) \cap C(\overline{R})$ to be a minimizer of the area functional in $\mathcal{A}_g$. Since, we are also assuming that $u \in \mathcal{A}_g$, we get that must solve the boundary value problem (BVP):

$$\left\{ \begin{array}{rcll} \nabla \cdot \left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) & = & 0 & \text{in } R; \\ u & = & g & \text{on } \partial R. \end{array} \right. \tag{2.114}$$

The BVP in (2.114) is called the **Dirichlet problem** for the minimal surface equation.

The PDE in (2.113) can also be written as

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0, \quad \text{in } R, \tag{2.115}$$

where the subscripted symbols read as follows:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y},$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2},$$

and

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = u_{yx}. \tag{2.116}$$

The fact that the "mixed" second partial derivatives in (2.116) are equal follows from the assumption that u is a C^2 function.

When we study the classification of PDEs we will see that the equation in (2.115) is a nonlinear, second order, elliptic PDE.

2.3.2 The Linearized Minimal Surface Equation

For the case in which the wire loop in the previous section is very close to a horizontal plane (see Figure 2.3.7), it is reasonable to assume that, if $u \in \mathcal{A}_g$,

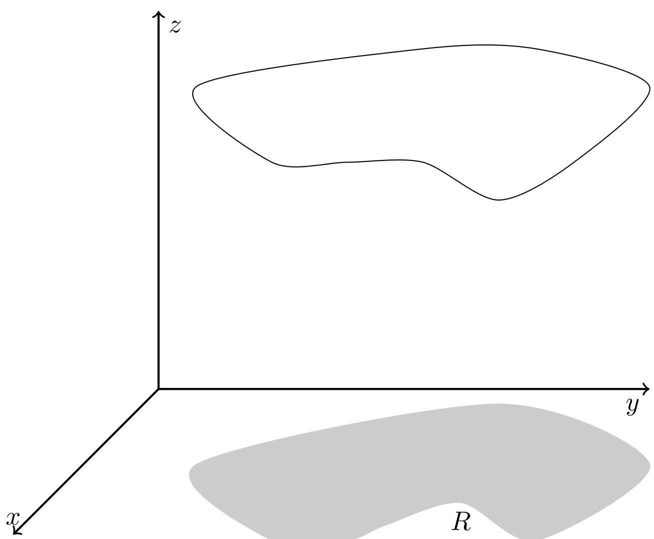

Figure 2.3.7: Almost Planar Wire Loop

$|\nabla u|$ is very small throughout R. We can therefore use the linear approximation

$$\sqrt{1+t} \approx 1 + \frac{1}{2}t, \quad \text{for small } |t|, \tag{2.117}$$

to approximate the area function in (2.100) by

$$A(u) \approx \iint_R \left[1 + \frac{1}{2}|\nabla u|^2\right]\, dxdy, \quad \text{for all } u \in \mathcal{A}_g,$$

so that

$$A(u) \approx \text{area}(R) + \frac{1}{2}\iint_R |\nabla u|^2\, dxdy, \quad \text{for all } u \in \mathcal{A}_g. \tag{2.118}$$

The integral on the right–hand side of the expression in (2.118) is known as the **Dirichlet Integral**. We will use it in these notes to define the Dirichlet functional, $\mathcal{D}\colon \mathcal{A}_g \to \mathbb{R}$,

$$\mathcal{D}(u) = \frac{1}{2}\iint_R |\nabla u|^2\, dxdy, \quad \text{for all } u \in \mathcal{A}_g. \tag{2.119}$$

Thus, in view of (2.118) and (2.119),

$$A(u) \approx \text{area}(R) + \mathcal{D}(u), \quad \text{for all } u \in \mathcal{A}_g. \tag{2.120}$$

Thus, according to (2.120), for wire loops close to a horizontal plane, minimal surfaces spanning the wire loop can be approximated by solutions to the following variational problem,

Problem 2.3.2 (Variational Problem 2)**.** *Out of all functions in $\mathcal{A}_g$, find one such that*

$$\mathcal{D}(u) \leqslant \mathcal{D}(v), \quad \textit{for all } v \in \mathcal{A}_g. \tag{2.121}$$

It can be shown that a necessary condition for $u \in \mathcal{A}_g$ to be a solution to the Variational Problem 2.3.2 is that u solves the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } R; \\ u = g & \text{on } \partial R, \end{cases} \tag{2.122}$$

where

$$\Delta u = u_{xx} + u_{yy},$$

the two–dimensional Laplacian (see Assignment #7). The BVP in (2.122) is called the Dirichlet Problem for Laplace's equation.

2.3.3 Vibrating String

Consider a string of length L (imagine a guitar string or a violin string) whose ends are located at $x = 0$ and $x = L$ along the x–axis (see Figure 2.3.8). We assume that the string is made of some material of (linear) density $\rho(x)$ (in units of mass per unit length). Assume that the string is fixed at the end–points and is tightly stretched so that there is a constant tension, τ, acting tangentially along the string at all times. We would like to model what happens to the

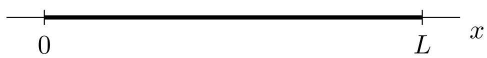

Figure 2.3.8: String of Length L at Equilibrium

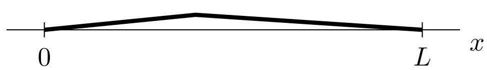

Figure 2.3.9: Plucked String of Length L

string after it is plucked to a configuration like that pictured in Figure 2.3.9 and then released. We assume that the shape of the plucked string is described by a continuous function, f, of x, for $x \in [0, L]$. At any time $t \geqslant 0$, the shape of the string is described by a function, u, of x and t; so that $u(x, t)$ gives the vertical displacement of a point in the string located at x when the string is in the equilibrium position pictured in Figure 2.3.8, and at time $t \geqslant 0$. We then have that

$$u(x, 0) = f(x), \quad \text{for all } x \in [0, L]. \tag{2.123}$$

In addition to the initial condition in (2.123), we will also prescribe the initial speed of the string,

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad \text{for all } x \in [0, L], \tag{2.124}$$

where g is a continuous function of x; for instance, if the plucked string is released from rest, then $g(x) = 0$ for all $x \in [0, L]$. We also have the boundary conditions,

$$u(0, t) = u(L, t) = 0, \text{ for all } t, \tag{2.125}$$

which model the assumption that the ends of the string do not move.

The question we would like to answer is: Given the initial conditions in (2.123) and (2.124), and the boundary conditions in (2.125), can we determine the shape of the string, $u(x, t)$, for all $x \in [0, L]$ and all times $t > 0$. We will answer this questions in a subsequent chapter in these notes. In this section, though, we will derive a necessary condition in the form of a PDE that u must satisfy in order for it to describe the motion of the vibrating string.

In order to find the PDE governing the motion of the string, we will formulate the problem as a variational problem. We will use Hamilton's Principle in Mechanics, or the **Principle of Least Action**. This principle states that the the path that configurations of a mechanical system take form time $t = 0$ to $t = T$ is such that a quantity called the **action** is minimized (or optimized) along the path. The action is defined by

$$A = \int_0^T [K(t) - V(t)]\, dt, \tag{2.126}$$

where $K(t)$ denotes the kinetic energy of the system at time t, and $V(t)$ its potential energy at time t. For the case of a string whose motion is described by small vertical displacements $u(x,t)$, for all $x \in [0,L]$ and all times t, the kinetic energy is given by

$$K(t) = \frac{1}{2}\int_0^L \rho(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2 \; dx. \tag{2.127}$$

To see how (2.127) comes about, note that the kinetic energy of a particle of mass m is

$$K = \frac{1}{2}mv^2,$$

where v is the speed of the particle. Thus, for a small element of the string whose projection on the x–axis is the interval $[x, x+\Delta x]$, so that its approximate length is Δx, the kinetic energy is, approximately,

$$\Delta K \approx \frac{1}{2}\rho(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2. \tag{2.128}$$

Thus, adding up the kinetic energies in (2.128) over all elements of the string adding in length to L, and letting $\Delta x \to 0$, yields the expression in (2.127), which we rewrite as

$$K(t) = \frac{1}{2}\int_0^L \rho u_t^2 \; dx, \quad \text{for all } t, \tag{2.129}$$

where u_t denotes the partial derivative of u with respect to t.

In order compute the potential energy of the string, we compute the work done by the tension, τ, along the string in stretching the string from its equilibrium length of L, to the length at time t given by

$$\int_0^L \sqrt{1+u_x^2}\; dx; \tag{2.130}$$

so that

$$V(t) = \tau\left[\int_0^L \sqrt{1+u_x^2}\; dx - L\right], \quad \text{for all } t. \tag{2.131}$$

Since we are considering small vertical displacements of the string, we can linearize the expression in in (2.130) by means of the linear approximation in (2.117) to get

$$\int_0^L \sqrt{1+u_x^2}\; dx \approx \int_0^L [1+\frac{1}{2}u_x^2]\; dx = L + \frac{1}{2}\int_0^L \frac{1}{2}u_x^2\; dx,$$

so that, substituting into (2.131),

$$V(t) \approx \frac{1}{2}\int_0^L \tau u_x^2\; dx, \quad \text{for all } t. \tag{2.132}$$

Thus, in view of, we consider the problem of optimizing the quantity

$$A(u) = \int_0^T \int_0^L \left[\frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2\right] \, dxdt, \tag{2.133}$$

where we have substitute the expressions for $K(t)$ and $V(t)$ in (2.129) and (2.132), respectively, into the expression for the action in (2.126).

We will use the expression for the action in (2.133) to the define a functional in the class of functions $\mathcal{A}$ defines as follows: Let $R = (0, L)\times(0, T)$, the cartesian product of the open intervals $(0, L)$ and $(0, T)$. Then, R is an open rectangle in the xt–plane. We say that $u \in \mathcal{A}$ if $u \in C^2(R) \cap C(\overline{R})$, and u satisfies the initial conditions in (2.123) and (2.124), and the boundary conditions in (2.125). Then, the action functional,

$$A : \mathcal{A} \to \mathbb{R},$$

is defined by the expression in (2.133), so that

$$A(u) = \frac{1}{2}\iint_R \left[\rho u_t^2 - \tau u_x^2\right] \, dxdt, \qquad \text{for } u \in \mathcal{A}. \tag{2.134}$$

Next, for $\varphi \in C_c^\infty(R)$, note that $u + s\varphi \in \mathcal{A}$, since φ has compact support in R, and therefore φ and all its derivatives are 0 on ∂R. We can then define a real valued function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(s) = A(u + s\varphi), \qquad \text{for } s \in \mathbb{R}, \tag{2.135}$$

Using the definition of the functional A in (2.134), we can rewrite $h(s)$ in (2.135) as

$$\begin{aligned} h(s) &= \frac{1}{2}\iint_R \left[\rho[(u + s\varphi)_t]^2 - \tau[(u + s\varphi)_x]^2\right] \, dxdt \\ &= \frac{1}{2}\iint_R \left[\rho[u_t + s\varphi_t]^2 - \tau[u_x + s\varphi_x]^2\right] \, dxdt, \end{aligned}$$

so that

$$h(s) = A(u) + s\iint_R [\rho u_t\varphi_t - \tau u_x\varphi_x] \, dxdt + s^2 A(\varphi), \tag{2.136}$$

for $s \in \mathbb{R}$, where we have used the definition of the action functional in (2.134).

It follows from (2.136) that h is differentiable and

$$h'(s) = \iint_R [\rho u_t\varphi_t - \tau u_x\varphi_x] \, dxdt + 2sA(\varphi), \qquad \text{for } s \in \mathbb{R}. \tag{2.137}$$

The principle of least action implies that, if u describes the shape of the string, then $s = 0$ must be ac critical point of h. Hence, $h'(0) = 0$ and (2.137) implies that

$$\iint_R [\rho u_t\varphi_t - \tau u_x\varphi_x] \, dxdt = 0, \qquad \text{for } \varphi \in C_c^\infty(R), \tag{2.138}$$

is a necessary condition for $u(x,t)$ to describe the shape of a vibrating string for all times t.

Next, we use the integration by parts formulas

$$\iint_R \psi \frac{\partial \varphi}{\partial x}\; dxdt = \int_{\partial R} \psi\varphi n_1\; ds - \iint_R \frac{\partial \psi}{\partial x}\varphi\; dxdt,$$

for C^1 functions ψ and φ, where n_1 is the first component of the outward unit normal, $\vec{n}$, on ∂R (wherever this vector is defined), and

$$\iint_R \psi \frac{\partial \varphi}{\partial t}\; dxdt = \int_{\partial R} \psi\varphi n_2\; ds - \iint_R \frac{\partial \psi}{\partial t}\varphi\; dxdt,$$

where n_2 is the second component of the outward unit normal, $\vec{n}$, (see Problem 1 in Assignment #8), to obtain

$$\iint_R \rho u_t \varphi_t\; dxdt = \int_{\partial R} \rho u_t \varphi n_2\; ds - \iint_R \frac{\partial}{\partial t}[\rho u_t]\varphi\; dxdt,$$

so that

$$\iint_R \rho u_t \varphi_t\; dxdt = -\iint_R \frac{\partial}{\partial t}[\rho u_t]\varphi\; dxdt, \tag{2.139}$$

since φ has compact support in R.

Similarly,

$$\iint_R \tau u_x \varphi_x\; dxdt = -\iint_R \frac{\partial}{\partial x}[\tau u_x]\varphi\; dxdt. \tag{2.140}$$

Next, substitute the results in (2.139) and (2.140) into (2.138) to get

$$\iint_R \left[\frac{\partial}{\partial t}[\rho u_t] - \frac{\partial}{\partial x}[\tau u_x]\right]\varphi\; dxdt = 0, \quad \text{for } \varphi \in C_c^\infty(R). \tag{2.141}$$

Thus, applying the result of Problem 2 in Assignment #6, we obtain from (2.141) that

$$\rho\frac{\partial^2 u}{\partial t^2} - \tau\frac{\partial^2 u}{\partial x^2} = 0, \quad \text{in } R, \tag{2.142}$$

since we area assuming that u is C^2, ρ is a continuous function of x, and τ is constant.

The PDE in (2.142) is called the one–dimensional **wave equation**. It is sometimes written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 u}{\partial x^2},$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{2.143}$$

where

$$c^2 = \frac{\tau}{\rho},$$

where the case in which ρ is assumed to be constant.

The wave equation in (2.142) or (2.143) is a second order, linear, hyperbolic PDE.

Chapter 3

How are PDEs Classified?

In the previous chapter we saw how various types of PDEs.

PDEs are classified according to order (the highest order of the derivative of the unknown functions involved in the equation). The Euler equations for an ideal, incompressible fluid,

$$\left\{\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho &= 0; \\ \rho\frac{\partial \vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} &= -\nabla p + \vec{f}; \\ \nabla\cdot\vec{u} &= 0, \end{aligned}\right. \tag{3.1}$$

are a system of first–order PDEs.

The 3–dimensional diffusion equation, or heat conduction equation,

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{3.2}$$

the two–dimensional Laplace's equation,

$$u_{xx} + u_{yy} = 0, \tag{3.3}$$

the one dimensional wave equations,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{3.4}$$

and the minimal surface equation,

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0, \tag{3.5}$$

are second order PDEs.

PDEs can further be classified as linear or nonlinear. For instance, the third equation in the system in (3.1), and the PDEs in (3.2), (3.3) and (3.4) are linear equations, while the first two equations in the system in (3.1) and the PDE in (3.5) are not linear. In the next section we will discuss properties of linear equations, and how those properties can be very helpful in the construction of solutions, and proofs of uniqueness for some initial/boundary value problems.

The PDE in (3.5) is in a general class of equations of the form

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d, \quad (3.6)$$

for some continuous function a, b, c and d of the five variables x, y, u, u_x and u_y, generally. If the coefficient functions in (3.6) depend only on x and y, we get the general linear second order equation in two variables,

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y). \quad (3.7)$$

If the coefficient functions in (3.6) do not depend on the derivatives of the unknown function u, we obtain the **quasi–linear**, second order PDE

$$a(x, y, u)u_{xx} + b(x, y, u)u_{xy} + c(x, y, u)u_{yy} = d(x, y, u). \quad (3.8)$$

In Section 3.2 we will discuss a further classification of the general second order PDE in (3.6) based on properties of certain curves associated with the equation known as **characteristic curves**. This will lead to the definition of **elliptic**, **hyperbolic** and **parabolic** PDEs. Laplace's equation,

$$u_{xx} + u_{yy} = 0, \quad (3.9)$$

the one–dimensional wave equation,

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0, \quad (3.10)$$

and the one–dimensional diffusion equations,

$$Du_{xx} - u_t = 0, \quad (3.11)$$

are archetypes of these classes of equations, respectively.

3.1 Linearity

Laplace's equation (3.9), the one–dimensional wave equation (3.10), and the one–dimensional diffusion equations (3.11) are linear equations. To understand the use of this terminology in the context of PDEs, note that Laplace's equation (3.9) can also be written as

$$\Delta u = 0,$$

where $\Delta \colon C^2(R) \to C(R)$ defines a linear operator between the spaces of functions $C^2(R)$ and $C(R)$ given by

$$\Delta u = u_{xx} + u_{yy}, \quad \text{for all } u \in C^(R), \quad (3.12)$$

for some open subset R of $\mathbb{R}^2$. The differential Δ defined in (3.12) is linear because of the linearity property of differentiation that we learned in Calculus; indeed, given functions $u, v \in C^2(R)$ and real constants c_1 and c_2, it follows from the linearity of differentiation that

$$\Delta(c_1 u + c_2 v) = c_1 \Delta u + c_2 \Delta v.$$

Similarly, the one–dimensional wave equation in (3.10) can be written as

$$-\Box u = 0,$$

where the linear operator $\Box\colon C^2(R) \to C(R)$, also known as the **d'Alembert operator**, is defined by

$$\Box u = \frac{1}{c^2} u_{tt} - u_{xx}, \quad \text{for all } u \in C^2(R),$$

where R is an open region in the xt–plane; and the one–dimensional diffusion equation in (3.11) can be written as

$$-Lu = 0,$$

where $L\colon C^2(R) \to C(R)$ defined by

$$Lu = u_t - D u_{xx}, \quad \text{for all } u \in C^2(R),$$

where R is an open region in the xt–plane, is also a linear operator.

By contrast, the map $N\colon C^1(R) \times C^1(R) \times C^1(R) \to C(R) \times C(R) \times C(R)$, given by

$$N(\vec{u}) = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}, \quad \text{for all } \vec{u} \in C^1(R) \times C^1(R) \times C^1(R),$$

where R is an open set in $\mathbb{R}^3$, is not linear (see Problem 1 in Assignment #9). Hence, the second PDE in the system (3.1) is not a linear equation.

In general, a linear PDE is an expression of the form

$$Lu = f, \tag{3.13}$$

where $L\colon \mathcal{U} \to \mathcal{F}$ is a linear differential operator from a linear space, $\mathcal{U}$, of differentiable functions to a linear space, $\mathcal{F}$, of continuous functions. An example of the equation in (3.13) is provided by Poisson's equation in Potential Theory,

$$-\Delta u = f, \quad \text{in } R, \tag{3.14}$$

where R is an open region in $\mathbb{R}^n$. In this case, the linear operator $L = -\Delta$ maps $C^2(R)$ to $C(R)$.

If $f = 0$ in the right–hand side of (3.13) we get the **homogeneous** PDE

$$Lu = 0. \tag{3.15}$$

The equation in (3.15) has the following very useful property known as the **principle of superposition**.

Proposition 3.1.1 (Principle of Superposition)**.** Let u and v denote two solutions of the homogeneous PDE in (3.15). Then, for any constants c_1 and c_2, the functions $c_1 u + c_2 v$ is also a solution of (3.15).

Proof: Since L is a linear differential operator, it follows that

$$L(c_1 u + c_2 v) = c_1 Lu + c_2 Lv.$$

Thus, if u and v solve (3.15), it follows that

$$L(c_1 u + c_2 v) = c_1 0 + c_2 0 = 0,$$

which shows that $c_1 u + c_2 v$ solves (3.15). ■

3.2 Classification of Second Order PDEs

Laplace's equation (3.9), the one–dimensional wave equation (3.10), and the one–dimensional diffusion equations (3.11) are second order PDEs. They are classified as **elliptic**, **hyperbolic** and **parabolic** PDEs, respectively. In this sections we study the rationale of that classification as it applies to the general second order PDE in two variables:

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d. \quad (3.16)$$

We begin with the special case of the linear equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y), \quad (3.17)$$

where a, b, c and d are continuous functions defined in some opens subset, R, of $\mathbb{R}^2$. The classification of the equations of the type in (3.16) or (3.17) is based on properties of curves in R associated with the equations; these curves are called **characteristic curves**. We begin with a curve, Γ, in R parametrized by a map $\gamma\colon I \to \mathbb{R}^2$,

$$t \mapsto \gamma(t) = (x(t), y(t)), \qquad \text{for } t \in I,$$

where I is some interval of real numbers; see Figure 3.2.1. Suppose we are trying to solve the linear PDE in (3.17) subject to information about u given on the curve Γ. Specifically, suppose we are given the values of u and its first derivatives on Γ; we can specify this conditions these condition on u by the equations

$$u(x(t), y(t)) = u_o(t), \quad \text{for } t \in I, \quad (3.18)$$

$$u_x(x(t), y(t)) = f(t), \quad \text{for } t \in I, \quad (3.19)$$

and

$$u_y(x(t), y(t)) = g(t), \quad \text{for } t \in I, \quad (3.20)$$

where u_o, f and g are given continuous functions on I. If we assume, in addition, that f and g are C^∞ functions, we can in theory obtain information about the

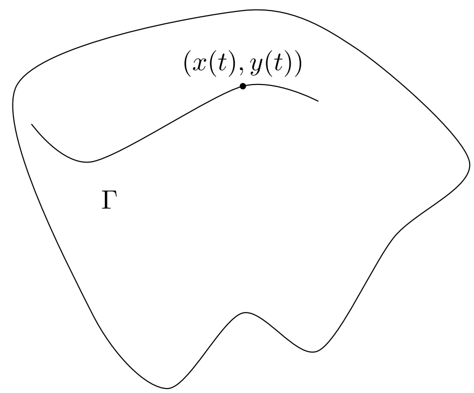

Figure 3.2.1: Characteristic Curves

second order derivatives, u_{xx}, u_{xy} and u_{yy}, of u (and higher order derivatives) on Γ. Is this can be done, we can attempt to construct a solution of the PDE in (3.17) by building Taylor series expansions around every point on Γ using the values of u and its derivatives based on the conditions in (3.18), (3.19) and (3.20) and derivatives of the expressions in (3.19) and (3.20). The first step in this construction consists of taking the derivatives of derivatives of the expressions in (3.19) and (3.20) and combining these with the information provided by the PDE (3.17) to obtain the linear system

$$\begin{cases} \dot{x}\, u_{xx} + \dot{y}\, u_{xy} = \dot{f} \\ \dot{x}\, u_{xy} + \dot{y}\, u_{yy} = \dot{g} \\ a\, u_{xx} + b\, u_{xy} + c\, u_{yy} = d, \end{cases} \tag{3.21}$$

for the unknowns u_{xx}, u_{xy} and u_{yy} on Γ, where a dot on top of a variable denotes derivative with respect to t:

$$\dot{x} = \frac{dx}{dt}, \ \dot{y} = \frac{dy}{dt}, \ \dot{f} = \frac{df}{dt} \ \text{ and } \ \dot{g} = \frac{dg}{dt}.$$

Note that the system in (3.21) can be written in matrix form as

$$\begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & b & c \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} \dot{f} \\ \dot{g} \\ d \end{pmatrix}. \tag{3.22}$$

The matrix equation in (3.22) can be solved for the second derivatives of u, in terms of the data $(\dot{f}, \dot{g}, d)$ on Γ, provided that the determinant of the matrix

$$\begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & b & c \end{pmatrix} \tag{3.23}$$

is not zero. The case in which the determinant of the matrix in (3.23) yields the equation

$$a(\dot{y})^2 - b\dot{x}\dot{y} + c(\dot{x})^2 = 0. \tag{3.24}$$

Dividing the equation in (3.24) by $\dot{x}^2$and using the fact that

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx},$$

by the Chain Rule, we obtain the ordinary differential equation (ODE)

$$a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0. \tag{3.25}$$

We shall refer to the ODE in (3.25) as the characteristic equation to the PDE in (3.17). Solutions to the ODE in (3.25) are called **characteristic curves** of the PDE in (3.17). Assuming that $a \neq 0$ in R, we can solve for $\dfrac{dy}{dx}$ in (3.25) to get

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.26}$$

We have three possibilities, depending on whether $b^2 - 4ac$ is positive, zero, or negative.

If $b^2 - 4ac > 0$, then the PDE in (3.17) has two families of characteristic curves given by the solutions to the two ODEs in (3.26). In this case we say that the PDE in (3.17) is **hyperbolic**.

If $b^2 - 4ac = 0$, then the PDE in (3.17) has one family of characteristic curves given by the solution to the ODE

$$\frac{dy}{dx} = \frac{b}{2a}.$$

In this case we say that the PDE in (3.17) is **parabolic**.

If $b^2 - 4ac < 0$, then the ODE in (3.25) has no real solutions. Thus, the PDE in (3.17) has no (real) characteristic curves. In this case we say that the PDE in (3.17) is **elliptic**.

Example 3.2.1 (The One–dimensional Wave Equation)**.** In the case of the linear second order equation

$$c^2 u_{xx} - u_{tt} = 0, \tag{3.27}$$

describing small amplitude vibrations of a string that we derived in Section 2.3.3 (see the PDE in (2.143), $a = c^2$, $b = 0$ and c in (3.25) is -1 (in this case t is playing the role of y). We therefore get that $b^2 - 4ac = -4(c^2)(-1) = 4c^2 > 0$; so that the equation in (3.27) is hyperbolic. For this PDE the equations for the characteristic curves in (3.26) yields

$$\frac{dt}{dx} = \pm\frac{2c}{2c^2},$$

or

$$\frac{dt}{dx} = \pm\frac{1}{c},$$

which we can rewrite as

$$\frac{dx}{dt} = \pm c. \tag{3.28}$$

The equations in (3.28) is a pair of ODEs that can be solved to yield the two families of curves

$$x = ct + \xi, \tag{3.29}$$

and

$$x = -ct + \eta, \tag{3.30}$$

where ξ and η are the parameters for each of the families of characteristic curves in (3.29) and (3.30). The family of characteristic curves described by the equations in (3.29) consists of parallel lines of (positive) slope $1/c$ in the xt–plane with x–intercept $\xi \in \mathbb{R}$. Some of those characteristic curves are shown in Figure 3.2.2.

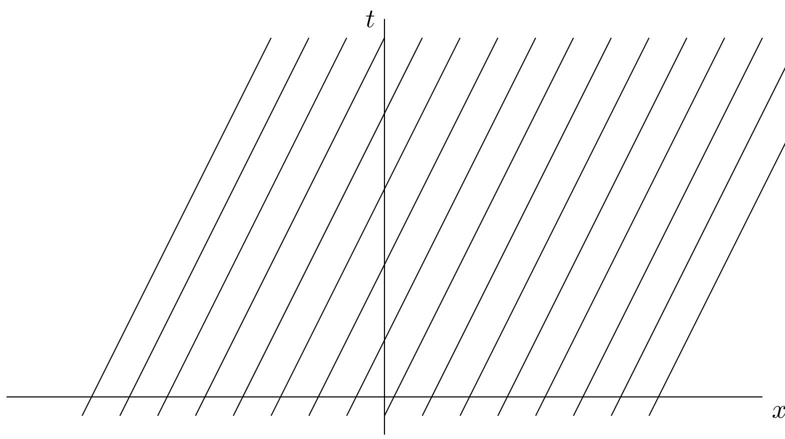

Figure 3.2.2: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

The family of characteristic curves described by the equations in (3.29) consist of parallel lines of (negative) slope $-1/c$ in the xt–plane and x–intercept $\eta\imath\mathbb{R}$; some of theses curves are sketched in Figure 3.2.3.

Figure 3.2.4 contains a sketch of both sets of characteristic curve in the same graph. We will see in Example 4.1.1 of Section 4.1 how to use the two sets of characteristic curves in Figure 3.2.4 to construct a solution to the initial value problem to the one–dimensional wave equation.

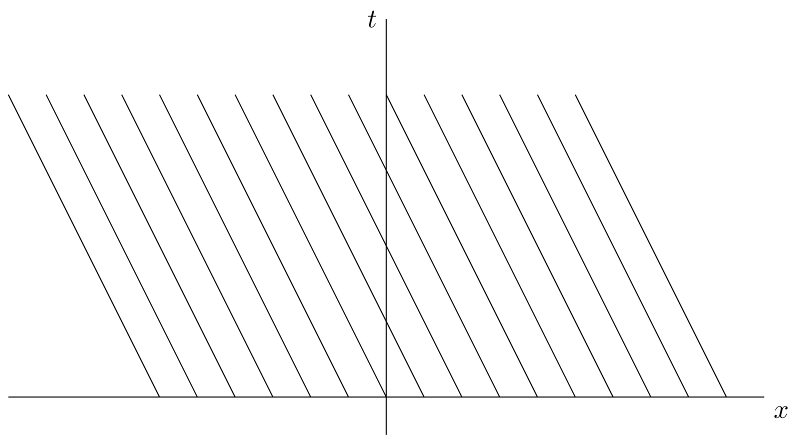

Figure 3.2.3: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

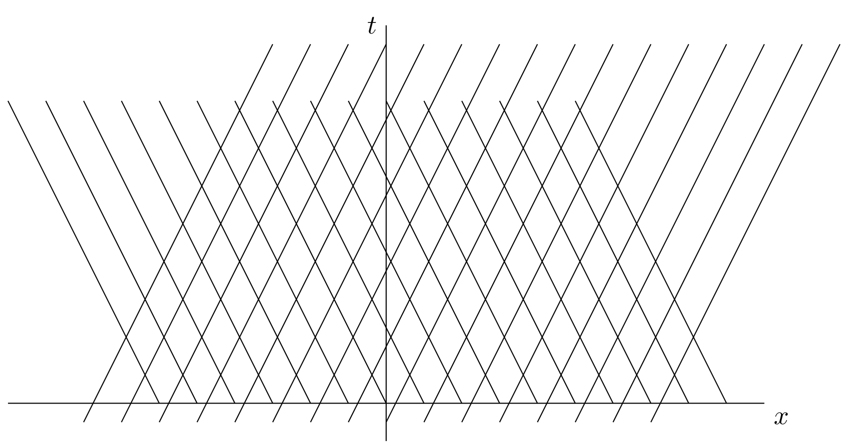

Figure 3.2.4: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

Chapter 4

How are PDEs Solved?

There is really no general theory for solving any given PDE of the form in (2.1),

$$F(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_n x_n}, \ldots) = 0.$$

Approaches to the construction of solutions of PDE problems are determined by the types of PDEs and the geometric properties (e.g., symmetries) of the equations and/or domains in which the problems are posed. In this chapter we present some of those approaches. Emphasis will be placed on a few general principles that can aid us when looking for solutions of PDEs.

We will begin with an approach that uses knowledge of the characteristic curves of the equations. We will then look at approaches that exploit any symmetries that the equations or domains in the problems might have. We will then look at methods of solutions for linear equations based on the principle of superposition.

4.1 Using Characteristic Curves to Solve PDEs

In Section 3.2 we defined characteristic curves for second order PDEs in two variables, and saw how characteristic curves can be used to come up with a classification scheme for those equations. In this section we see how to use characteristic curves to construct solutions to certain types of PDEs in two variables. We begin with the example of the one–dimensional wave equation.

4.1.1 Solving the One–Dimensional Wave Equation

Example 4.1.1 (Solving the One–Dimensional Wave Equation)**.** We consider the initial value problem in the entire real:

$$\begin{cases} u_{xx} - \dfrac{1}{c^2} u_{tt} = 0, & \text{for } x \in \mathbb{R}, t > 0, \\ \\ u(x, 0) = f(x), & \text{for all } x \in [0, L]; \\ u_t(x, 0) = g(x), & \text{for all } x \in [0, L], \end{cases} \tag{4.1}$$

where f and g are given continuous functions defined in $\mathbb{R}$.

In Example 3.2.1 in the previous section we showed that the PDE in (4.1) has two families of characteristic curves given by

$$x = ct + \xi, \tag{4.2}$$

and

$$x = -ct + \eta. \tag{4.3}$$

These families of curves consist of parallel straight lines in the xt–plane of slope $1/c$ and of slope $-1/c$, respectively. We will see in the next section that characteristic curves carry information about solutions of the equation from one point on the curve to another point on the same curve. This suggests that we we consider the PDE in (4.1) along the curves given in (4.2) and (4.3). We can do this my considering the parameters ξ and η in (4.2) and (4.3) as a new set of variables,

$$\xi = x - ct, \tag{4.4}$$

and

$$\eta = x + ct. \tag{4.5}$$

If we are give a solution, u, to the PDE in (4.1), we can use the change of variables provided by (4.4) and (4.5) to define a function, v, of ξ and η in terms of u by means of

$$v(\xi, \eta) = u(x, t), \tag{4.6}$$

where x and t can be obtained in terms of ξ and θ by solving the linear system

$$\begin{cases} x - ct &= \xi; \\ x + ct &= \eta, \end{cases}$$

so that

$$\begin{aligned} x &= \frac{1}{2}\eta + \frac{1}{2}\xi; \\ t &= \frac{1}{2c}\eta - \frac{1}{2c}\xi. \end{aligned} \tag{4.7}$$

Alternatively, we can rewrite (4.6) as

$$u(x, t) = v(\xi, \eta), \tag{4.8}$$

where ξ and η are given by (4.4) and (4.5), respectively.

Assume that $u \in C^2(\mathbb{R}^2)$ solves the PDE in (4.1). We would like to derive a PDE satisfied by the function v defined in (4.6) and (4.7). The PDE that v will satisfy will be expressed in terms of the new variables ξ and η. In order to do this, we use (4.8) and the Chain Rule to get

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x}, \tag{4.9}$$

where

$$\frac{\partial \xi}{\partial x} = 1 \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 1, \tag{4.10}$$

by virtue of (4.4) and (4.5). Thus, substituting the expressions in (4.10) into (4.9),

$$u_x = v_\xi + v_\eta. \tag{4.11}$$

Next, take partial derivative with respect to x on both sides of (4.11) and use the Chain Rule to get

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}[v_\xi] + \frac{\partial}{\partial x}[v_\eta] \\ &= v_{\xi\xi}\frac{\partial \xi}{\partial x} + v_{\xi\eta}\frac{\partial \eta}{\partial x} + v_{\eta\xi}\frac{\partial \xi}{\partial x} + v_{\eta\eta}\frac{\partial \eta}{\partial x}, \end{aligned}$$

so that, using the expressions in (4.10) and the fact that mixed partial derivatives, $v_{\xi\eta}$ and $v_{\eta\xi}$, of C^2 functions are equal,

$$u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}. \tag{4.12}$$

Similar calculations to those leading to (4.12) can be used to obtain an expression for u_{tt}. Indeed, take partial derivative with respect to t on both sides of (4.8) and use the Chain Rule to get

$$u_t = v_\xi \frac{\partial \xi}{\partial t} + v_\eta \frac{\partial \eta}{\partial t}, \tag{4.13}$$

where

$$\frac{\partial \xi}{\partial x} = -c \quad \text{and} \quad \frac{\partial \eta}{\partial x} = c, \tag{4.14}$$

by virtue of (4.4) and (4.5). Thus, substituting the expressions in (4.14) into (4.13),

$$u_t = -cv_\xi + cv_\eta. \tag{4.15}$$

Next, take partial derivative with respect to t on both sides of (4.15) and use the Chain Rule to get

$$\begin{aligned} u_{tt} &= \frac{\partial}{\partial t}[v_\xi] + \frac{\partial}{\partial t}[v_\eta] \\ &= -cv_{\xi\xi}\frac{\partial \xi}{\partial t} - cv_{\xi\eta}\frac{\partial \eta}{\partial t} + cv_{\eta\xi}\frac{\partial \xi}{\partial t} + cv_{\eta\eta}\frac{\partial \eta}{\partial t}; \end{aligned}$$

thus, using the expressions in (4.14), we get

$$u_{tt} = c^2 v_{\xi\xi} - c^2 v_{\xi\eta} - c^2 v_{\eta\xi} + c^2 v_{\eta\eta},$$

or

$$u_{tt} = c^2[v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}], \tag{4.16}$$

where we have used the equality of the mixed second partial derivatives.

Since we are assuming that u solves the PDE

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0, \tag{4.17}$$

it follows from (4.12), (4.16) and (4.17) that v solves the second order PDE

$$v_{\xi\eta} = 0. \tag{4.18}$$

Note that the PDE in (4.18) is also a hyperbolic, second order, linear PDE (in this case $a = c = 0$ and $b = 1$, so that $b^2 - 4ac = 1 > 0$). In contrasts with the hyperbolic PDE in (4.17), the PDE in (4.18) can be solved directly by integration. Indeed, writing (4.18) as

$$\frac{\partial}{\partial \eta}[v_\xi] = 0,$$

we see that

$$v_\xi = h(\xi), \tag{4.19}$$

where h is an arbitrary C^1 function of a single variable; and writing (4.19) as

$$\frac{\partial}{\partial \xi}[v(\xi, \eta)] = 0,$$

we see that

$$v(\xi, \eta) = F(\xi) + G(\eta), \tag{4.20}$$

where F is an antiderivative of h (i.e., $F' = h$), and G is an arbitrary C^2 function of a single variable.

The function v defined by the expression in (4.20), where F and G are arbitrary C^2 functions of a single variable, is the **general solution** to the PDE in (4.18). We can use it, along with (4.8), (4.4) and (4.5) to obtain the general solution to the one–dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad \text{for } x \in \mathbb{R}, \quad \text{and } t \in \mathbb{R}; \tag{4.21}$$

namely,

$$u(x, t) = F(x - ct) + G(x + ct), \tag{4.22}$$

where F and G are arbitrary C^2 functions of a single variable. The expression in (4.22) is known as d'Alembert's solutions to the one–dimensional wave equation.

We now use the general solution (4.22) to the one–dimensional wave equation in (4.21) to construct a solution to the initial value problem in (4.1). In this construction we will need to assume that f is C^2 and g is C^1.

Differentiate the expression for u in (4.22) with respect to t to obtain

$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct), \tag{4.23}$$

where we have applied the Chain Rule. Next, apply the initial conditions in (4.1) to obtain the equations

$$\left\{\begin{array}{rcl} F(x) + G(x) & = & f(x); \\ -cF'(x) + cG'(x) & = & g(x), \end{array}\right. \tag{4.24}$$

for all $x \in \mathbb{R}$, where we have used (4.22) and (4.23).

Next, differentiate the first equation in (4.24) and divide the second equation by c to get

$$\left\{\begin{array}{rcl} F'(x) + G'(x) & = & f'(x); \\ -F'(x) + G'(x) & = & g(x)/c, \end{array}\right. \tag{4.25}$$

for all $x \in \mathbb{R}$, where we have used the differentiability assumptions on f.

Adding the equations in (4.25) then yields the following equation for G':

$$G'(x) = f'(x) + \frac{1}{2c}g(x), \quad \text{for all } x \in \mathbb{R}. \tag{4.26}$$

Integrating the equation in (4.26) then yields

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\,dz + C_1, \quad \text{for all } x \in \mathbb{R}, \tag{4.27}$$

where C_1 is a constant of integration.

Similarly, subtracting the second equation in (4.25) from the first equation and integrating yields

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\,dz + C_2, \quad \text{for all } x \in \mathbb{R}, \tag{4.28}$$

where C_2 is a constant of integration.

Next, substitute the functions in (4.28) and (4.27) into the formula for $u(x,t)$ in (4.22) to get

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz + C_3, \tag{4.29}$$

for all $x \in \mathbb{R}$, where $C_3 = C_1 + C_2$.

It follows from the first initial condition in (4.1) that the constant C_3 in (4.29) must be 0, so that

$$u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz, \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}, \tag{4.30}$$

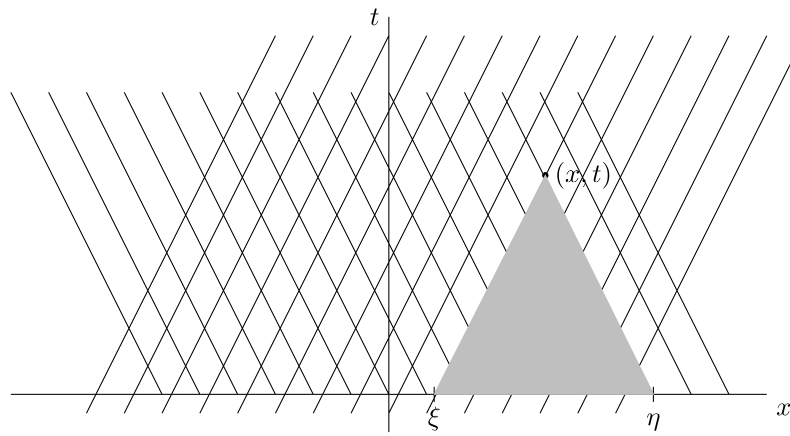

Figure 4.1.1: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

solves the initial value problem (4.1) for the one–dimensional wave equation.

In order to understand what the solution to the IVP in (4.1) displayed in (4.30) is saying, refer to Figure 4.1.1. Suppose we want to compute the value of u at x and at time $t > 0$; that is, $u(x,t)$, where (x,t) is a point in the xt–plane. Two characteristic curves cross at that point: one with x–intercept labeled ξ in Figure 4.1.1, and the other with x–intercept labeled η in the figure. These correspond to the values $x - ct$ and $x + ct$, respectively. According to the expression for u in (4.29), the value of u at (x,t) is the average the values of the initial data f at those two points, plus t times the average of all the values of the initial speed, g, over the interval $[\xi, \eta]$.

For the special case in which the initial speed is zero throughout $\mathbb{R}$, we obtain from (4.30) the special form

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)], \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}. \tag{4.31}$$

The function u in (4.31) is made up of two traveling wave forms: $\frac{1}{2}f(x - ct)$, which moves to the right with speed c, and $\frac{1}{2}f(x + ct)$, which moves to the left with speed c. We illustrate this for the spacial case in which the initial data f is in $C_c^\infty(\mathbb{R})$, with $\text{supp}(f) = [-1, 1]$; see Figure 4.1.2. Figure 4.1.3 shows the supports of the initial data and two of the traveling waves at some time $t > 0$ later with $ct > 2$. Figure 4.1.4 shows the two pulses traveling in opposite directions at that instant of time. Note that the two pulses in Figure 4.1.4 have half of the amplitude of the initial pulse in Figure 4.1.2.

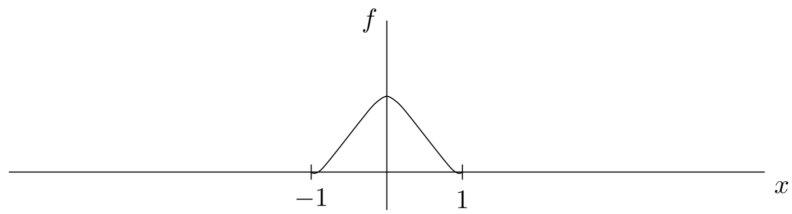

Figure 4.1.2: Initial Data

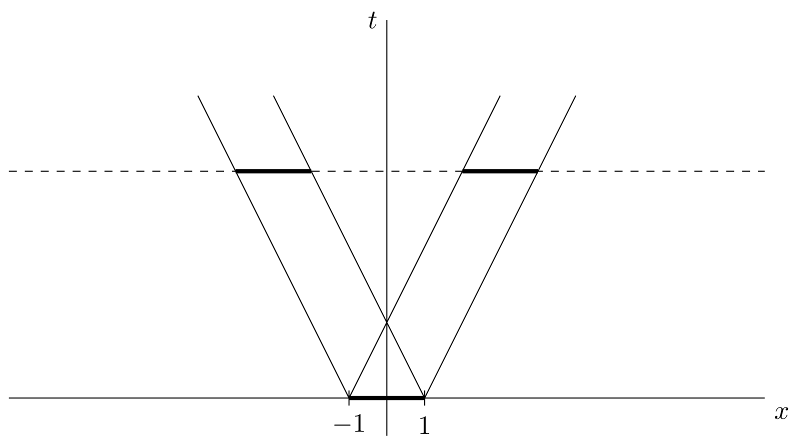

Figure 4.1.3: Traveling Waves

4.1.2 Solving First–Order PDEs

In this section we define characteristic curves for the first order equation in two variables

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u), \tag{4.32}$$

where a, b and c are C^∞ functions of three real variables, (x,y,z), where (x,y) lies in an open region, R, in $\mathbb{R}^2$. For the case in which coefficient functions, a, b and c, in (4.32) depend only on $(x,y) \in R$, the PDE in (4.32) turns into the linear PDE:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y), \quad \text{for } (x,y) \in R. \tag{4.33}$$

We will first define the concept of characteristic curves for the PDE in (4.33). The discussion here is analogous to the discussion on characteristic curves for the second order equation in (3.17) on page 42. As in that discussion, the

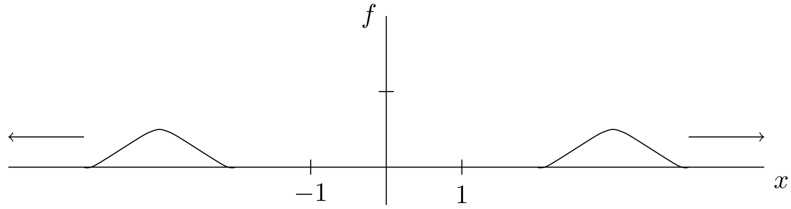

Figure 4.1.4: Traveling Pulses

starting point is smooth curve, Γ, in R parametrized by a map $\gamma\colon I \to \mathbb{R}^2$,

$$t \mapsto \gamma(t) = (x(t), y(t)), \qquad \text{for } t \in I,$$

where I is some interval of real numbers; see Figure 3.2.1. Suppose we are trying to solve the linear PDE in (4.33) subject to an "initial" condition on the curve Γ given by

$$u(x(t), y(t)) = f(t), \quad \text{for } t \in I, \tag{4.34}$$

where f is a known smooth function defined on I. The idea is that, given the information in (4.34), we can use that information together with the PDE in (4.33), to obtain the values of the derivatives, u_x and u_y, of u on Γ. Once these are obtained, we can differentiate (4.34) and the PDE in (4.33) to obtain information of the second derivatives on Γ. Since we are assuming that the coefficients, a, b and c, and the "initial" data, f, are C^∞ functions, we can, in theory, proceed in this fashion to obtain information about the higher order derivatives of u on Γ. If this can be done, we can attempt to construct a solution of the PDE in (4.33) by building Taylor series expansions around every point on Γ using the values of u and its derivatives. The first step in this construction is possible provided that the linear system

$$\begin{cases} \dot{x}\, u_x \ + \ \dot{y}\, u_y = \dot{f} \\ a\, u_x \ + \ b\, u_y = c, \end{cases} \tag{4.35}$$

for the unknowns u_x, and u_y on Γ can be solved. The system in (4.35) can be written in matrix form as

$$\begin{pmatrix} \dot{x} & \dot{y} \\ a & b \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \dot{f} \\ c \end{pmatrix}. \tag{4.36}$$

The matrix equation in (4.36) can be solved for the first derivatives of u, in terms of the data $\dot{f}$ on Γ, provided that the determinant of the matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \\ a & b \end{pmatrix} \tag{4.37}$$

is not zero. The case in which the determinant of the matrix in (4.37) is zero yields the equation for the characteristic curves of the first order PDE in (4.33):

$$b\dot{x} - a\dot{y} = 0. \tag{4.38}$$

Observe that the ODE in (4.38) is equivalent to the system of first order ODEs:

$$\begin{cases} \dfrac{dx}{dt} & = & a(x,y); \\ \\ \dfrac{dy}{dt} & = & b(x,y). \end{cases} \tag{4.39}$$

The system of ODEs in (4.39) defines the characteristic curves for the first–order linear PDE in (4.33). Since, we are assuming that a and b ar C^∞ functions, solutions to the system of first–order ODEs in (4.39) is guaranteed to have a unique solution around $t_o \in \mathbb{R}$ subject to the the initial condition $(x(t_o), y(t_o)) = (x_o, y_o)$. Thus, in theory, characteristic curves for the PDE in (4.33) can always be computed.

Suppose that we have computed the characteristic curves for the PDE in (4.33) according to the system of ODEs in (4.39). Let one of those characteristics be given by the parametrization

$$t \mapsto (x(t), y(t)), \qquad \text{for } t \in I, \tag{4.40}$$

where I is a maximal interval of existence. Suppose that u is a solution of the PDE in (4.32) and consider the values of u on the characteristic curve parametrized by (4.40),

$$u(x(t), y(t)), \qquad \text{for } t \in I. \tag{4.41}$$

It follows from (4.41) and the Chain Rule that

$$\frac{d}{dt}[u(x(t), y(t))] = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt},$$

so that, using the definition of the characteristic curves in (4.39),

$$\frac{d}{dt}[u(x(t), y(t))] = a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y),$$

by virtue of the PDE in (4.33). We have therefore shown that if u is a solution of the PDE in (4.33), then u must also solve the ODE

$$\frac{du}{dt} = c(x,y) \tag{4.42}$$

along the characteristic curves. This suggests a way to construct a solution to initial value problem for the PDE in (4.33) where the initial data is given on a curve that is not a characteristic curve. This approach is known as the **method of characteristic curves**.

Suppose we want to solve the IVP:

$$\begin{cases} a(x,y)u_x + b(x,y)u_y & = & c(x,y) \quad \text{in } R; \\ \\ u & = & f \quad \text{on } \Gamma, \end{cases} \tag{4.43}$$

where Γ is a curve in R that is not a characteristic curve. The method of characteristic curves consists of, first, finding the characteristic curves of the PDE in (4.43) by solving the system of ODEs in (4.39). Then, solve the ODE in (4.42). We illustrate this method by solving the following IVP for the one–dimensional **convection equation**.

Example 4.1.2 (One–Dimensional Convection Equation)**.** Consider the IVP

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 & \text{for } x \in \mathbb{R} \text{ and } t > 0; \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{4.44}$$

where c is a nonzero constant and f is a given C^1 function defined in $\mathbb{R}$.

In this example t is playing the role of y, so that the equations for the characteristic curves in (4.39) become the single ODE

$$\frac{dx}{dt} = c, \tag{4.45}$$

which can be solved to yield

$$x = ct + \xi, \tag{4.46}$$

where ξ is a real parameter indexing the characteristic curves. For the case in which $c > 0$ the characteristic curves for the PDE in (4.44) are straight lines of positive slope $1/c$ in the xt–plane and x–intercept ξ. Some of these curves are sketched in Figure 4.1.5. Along each characteristic curve in (4.46), a solution

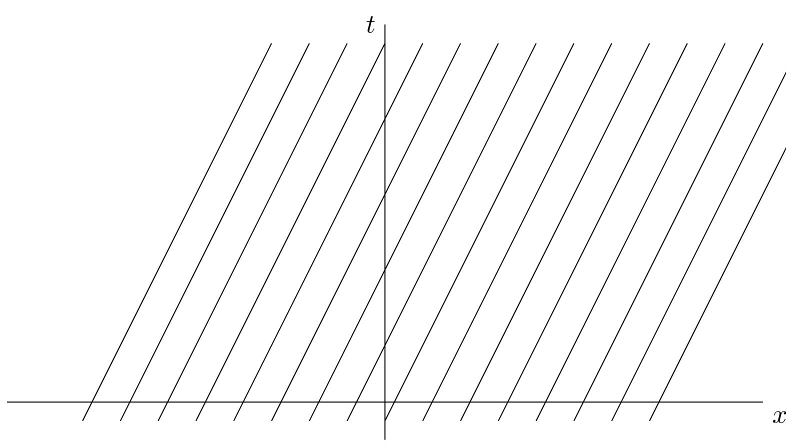

Figure 4.1.5: Characteristic Curves of $u_t + cu_x = 0$

to the PDE in (4.44) solves the ODE

$$\frac{du}{dt} = 0, \tag{4.47}$$

according to (4.42). Alternatively, we can obtain (4.47) by computing

$$\begin{aligned}\frac{d}{dt}[u(x(t),t)] &= u_x \cdot \frac{dx}{dt} + u_t \\ &= u_t + cu_x \\ &= 0,\end{aligned}$$

where we have used the Chain Rule, (4.45), and the assumption that u solves the PDE in (4.44).

We can solve the ODE in (4.47) to obtain that

$$u(x,t) = \text{constant along characteristic curves} \tag{4.48}$$

Since the characteristic curves in (4.46) are indexed by ξ, we can rewrite (4.48) as

$$u(x,t) = F(\xi), \tag{4.49}$$

where F is an arbitrary C^1 function of a real variable. Next, solve for ξ in (4.46) and substitute into (4.49) to obtain the general solutions to the PDE in (4.44),

$$u(x,t) = F(x - ct), \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}. \tag{4.50}$$

For the case in which $c > 0$, (4.50) describes a traveling wave moving to the right with speed c.

Finally, using the initial condition in (4.44), we get that

$$F(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that

$$u(x,t) = f(x - ct), \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R},$$

is a solution to the IVP in (4.44).

The method of characteristic curves illustrated thus far also applied to the quasi–linear equation in (4.32). In this case, the equations to the characteristic curves read

$$\left\{\begin{aligned}\frac{dx}{dt} &= a(x,y,u); \\ \frac{dy}{dt} &= b(x,y,u).\end{aligned}\right. \tag{4.51}$$

Along characteristic curves u solves the ODE

$$\frac{du}{dt} = c(x,y,u). \tag{4.52}$$

In general, we might not be able to obtain an explicit formula for a solution of the PDE in (4.32) based on the system (4.50)–(4.52). But, in some cases, we might be able to obtain an expression that gives $u(x,y)$ implicitly. We illustrate this in the following example.

Example 4.1.3. Find a solution to the initial value problem

$$\begin{cases} u_t + uu_x = 0, & \text{for } x \in \mathbb{R}, t > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{4.53}$$

Here t is playing the role of y in the general discussion. In this case, the equation for the characteristic curves is

$$\frac{dx}{dt} = u. \tag{4.54}$$

In order to solve the ODE in (4.54) we need information on the function u, which is what ultimately we are trying too determine. The information is provided by the differential equation that u satisfies along characteristic curves; namely,

$$\frac{du}{dt} = 0,$$

which implies that u must be constant along characteristic curves. Thus, we can set

$$u = F(\xi), \tag{4.55}$$

where ξ is a parameter indexing the characteristic curves, and F is an arbitrary C^1 function of a single variable. Substituting the expression for u in (4.55) into the equation for the characteristic curves in (4.54) yields

$$\frac{dx}{dt} = F(\xi),$$

which can be solved to yield the equation for the characteristic curves of the PDE in (4.53):

$$x = F(\xi)t + \xi. \tag{4.56}$$

Observe that in this case the characteristic curves are straight lines in the xt–plane with x–intercept ξ and slope $1/F(\xi)$. Note that the slopes of the characteristic curves depend on the value of the solution on the characteristic curves, according to (4.55).

We can solve for ξ in (4.56) and use (4.55) to get

$$\xi = x - u(x,t)t$$

and then substitute this value into (4.55) to obtain an implicit formula for $u(x,t)$:

$$u(x,t) = F(x - u(x,t)t), \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0. \tag{4.57}$$

Next, use the initial condition in (4.53) to obtain from (4.57) that

$$F(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that

$$u(x,t) = f(x - u(x,t)t), \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

provides an expression that defines $u(x,t)$ implicitly.

In the remainder of this section, we present more examples on the use of characteristic curves to solve first order PDEs.

Example 4.1.4. Find a solution to the initial value problem

$$\begin{cases} u_t + u_x = u, & \text{for } x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{4.58}$$

where f is a given C^1 function.

The equation for the characteristic curves in this example is

$$\frac{dx}{dt} = 1,$$

which can be solved to yield

$$x = t + \xi. \tag{4.59}$$

Now, along characteristic curves, u solves the ODE

$$\frac{du}{dt} = u;$$

so that

$$u = F(\xi)e^t, \tag{4.60}$$

where F is a C^1 function of a real variable, and ξ is the parameter indexing the characteristic curves in (4.59).

Next, solve for ξ in (4.59) and substitute into (4.60) to get the general solution,

$$u(x,t) = F(x-t)e^t, \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0, \tag{4.61}$$

for the PDE in (4.58), where F is an arbitrary C^1 function. The initial condition in (4.58) can now be used to obtain from (4.61) that

$$F(x) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

It then follows from (4.61) that

$$u(x,t) = f(x-t)e^t, \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

solves the initial value problem in (4.58).

Example 4.1.5. Find the general solution to the linear partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{4.62}$$

The equations for the characteristic curves are

$$\begin{cases} \dfrac{dx}{dt} = x; \\[2ex] \dfrac{dy}{dt} = y. \end{cases} \tag{4.63}$$

Using the Chain Rule, we obtain from (4.63) the ODE

$$\frac{dy}{dx} = \frac{y}{x}, \quad \text{for } x \neq 0. \tag{4.64}$$

The ODE in (4.64) can be solved by separating variables to yield

$$y = \xi x, \tag{4.65}$$

where ξ is a real parameter. Thus, the characteristic curves for the PDE in (4.62) is a pencil of straight lines through the origin in $\mathbb{R}^2$.

Now, along the characteristic curves for the PDE in (4.62), u solves the ODE

$$\frac{du}{dt} = 2u. \tag{4.66}$$

Next, combine the ODE in (4.66) with the first ODE in (4.63) to obtain the ODE

$$\frac{du}{dx} = \frac{2u}{x}, \quad \text{for } x \neq 0. \tag{4.67}$$

The ODE in (4.67) can be solved by separation of variables to yield

$$u = F(\xi)x^2, \tag{4.68}$$

where F is an arbitrary C^1 function, and ξ is the parameter indexing the characteristic curves in (4.65).

Solving for ξ in (4.65) and substituting into (4.68) then yields the general solution,

$$u(x, y) = F\left(\frac{y}{x}\right)x^2, \quad \text{for } x \neq 0.$$

4.2 Using Symmetry to Solve PDEs

A partial differential equation is said to be **invariant** under a group of transformations if its form does not change after a changing variables according to the transformations in the group. We illustrate this idea by looking at symmetric solutions to Laplace's equation in $\mathbb{R}^2$.

4.2.1 Radially Symmetric Solutions to Laplace's Equation

Suppose that u is a C^2 solution to Laplace's equation in $\mathbb{R}^2$,

$$u_{xx} + u_{yy} = 0. \tag{4.69}$$

We consider what happens to the equation in (4.69) when we change to a new set of variables, $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, given by a one–parameter group of rotations given by the matrices

$$M_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}; \tag{4.70}$$

that is, rotations in the counterclockwise sense by an angle ϕ. We set

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = M_\phi \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.71}$$

or

$$\left\{ \begin{array}{rcl} \xi & = & x\cos\phi - y\sin\phi; \\ \eta & = & x\sin\phi + y\cos\phi, \end{array} \right. \tag{4.72}$$

in view of (4.70) and (4.71). The equations in (4.72) can be solved for x and y in terms of ξ and η by inverting the matrix in (4.70),

$$M_\phi^{-1} = M_{-\phi} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$

so that

$$\left\{ \begin{array}{rcl} x & = & \xi\cos\phi + \eta\sin\phi; \\ y & = & -\xi\sin\phi + \eta\cos\phi. \end{array} \right. \tag{4.73}$$

In view of the equations in (4.73), we can think of u as a function of ξ and η, which we will denote by $v(\xi,\eta)$; so that

$$v(\xi,\eta) = u(x,y), \tag{4.74}$$

where x and y on the right–hand side of (4.74) are given in terms of ξ and η in (4.73).

Applying the Chain Rule, we obtain from (4.74) that

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x},$$

where

$$\frac{\partial \xi}{\partial x} = \cos\phi \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \sin\phi, \tag{4.75}$$

in view of the equations in (4.72). Thus,

$$u_x = \cos\phi\ v_\xi + \sin\phi\ v_\eta. \tag{4.76}$$

Similar calculations using (4.74) and (4.72) yield

$$u_y = -\sin\phi\ v_\xi + \cos\phi\ v_\eta. \tag{4.77}$$

Next, differentiate on both sides of (4.76) with respect to x and apply the Chain Rule to get

$$u_{xx} = \cos\phi\left[v_{\xi\xi}\frac{\partial \xi}{\partial x} + v_{\xi\eta}\frac{\partial \eta}{\partial x}\right] + \sin\phi\left[v_{\eta\xi}\frac{\partial \xi}{\partial x} + v_{\eta\eta}\frac{\partial \eta}{\partial x}\right],$$

so that, using (4.75) and the fact that the mixed second partial derivatives of C^2 functions are equal,

$$u_{xx} = \cos^2\phi\ v_{\xi\xi} + 2\sin\phi\cos\phi\ v_{\xi\eta} + \sin^2\phi\ v_{\eta\eta}. \tag{4.78}$$

Similarly, taking the partial derivative with respect to y on both sides of (4.77), and using

$$\frac{\partial \xi}{\partial y} = -\sin\phi \quad \text{and} \quad \frac{\partial \eta}{\partial y} = \cos\phi,$$

which follow from (4.72), we obtain from (4.77) that

$$u_{yy} = \sin^2\phi\, v_{\xi\xi} - 2\sin\phi\cos\phi\, v_{\xi\eta} + \cos^2\phi\, v_{\eta\eta}. \tag{4.79}$$

Thus, adding the expressions in (4.78) and (4.79),

$$u_{xx} + u_{yy} = v_{\xi\xi} + v_{\eta\eta}.$$

Hence, if u solves Laplace's equation in (4.69), then v solves the equation

$$v_{\xi\xi} + v_{\eta\eta} = 0,$$

which has the same form as Laplace's equation. We therefore conclude that Laplace's equation in (4.69) is invariant under rotations. This suggests that we look for solutions of (4.69) that are functions of a combination of the independent variables that is independent of the rotation parameter ϕ. To obtain such a combination, use (4.72) to compute

$$\begin{aligned} \xi^2 + \eta^2 &= (x\cos\phi - y\sin\phi)^2 + (x\sin\phi + y\cos\phi)^2 \\ &= x^2\cos^2\phi - 2xy\cos\phi\sin\phi + y^2\sin^2\phi \\ &\quad + x^2\sin^2\phi + 2xy\sin\phi\cos\phi + y^2\cos^2\phi \\ &= x^2 + y^2, \end{aligned}$$

so that x^2+y^2 or $\sqrt{x^2+y^2}$ are combinations of the independent variables, x and y, that do not depend on ϕ, the rotation parameter; that is, they are rotationally invariant. We will therefore look for solutions of the Laplace's equation in (4.69) that are of the form

$$u(x,y) = f(\sqrt{x^2+y^2}), \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{4.80}$$

where f is a C^2 function of a single variable. Functions of the form in (4.80) are said to be **radially symmetric**.

Example 4.2.1 (Radially Symmetric Solutions of Laplace's Equation in $\mathbb{R}^2$)**.** Let $R = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$. Find all radially symmetric solutions of (4.69) in R.

Solution: We look for solutions of

$$u_{xx} + u_{yy} = 0, \quad \text{in } R, \tag{4.81}$$

of the form

$$u(x,y) = f(r), \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{4.82}$$

where

$$r = \sqrt{x^2 + y^2}, \tag{4.83}$$

and $f\colon (0, \infty) \to \mathbb{R}$ is a C^2 function.

Write the expression in (4.83) $r^2 = x^2 + y^2$ and differentiate on both sides with respect to x, applying the Chain Rule to get

$$2r\frac{\partial r}{\partial x} = 2x,$$

from which we get that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \text{for } r > 0. \tag{4.84}$$

Similar calculations show that

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{for } r > 0, \tag{4.85}$$

Next, use the Chain Rule to obtain from (4.82) that

$$u_x = f'(r)\frac{\partial r}{\partial x},$$

so that, by virtue of (4.84),

$$u_x = \frac{x}{r}f'(r), \quad \text{for } r > 0. \tag{4.86}$$

Similar calculations using (4.82) and (4.85) yield

$$u_y = \frac{y}{r}f'(r), \quad \text{for } r > 0. \tag{4.87}$$

Next, use the Product Rule, the Quotient Rule, and the Chain Rule to obtain from (4.86) that

$$u_{xx} = \frac{1}{r}f'(r) + x\frac{rf''(r)\dfrac{\partial r}{\partial x} - f'(r)\dfrac{\partial r}{\partial x}}{r^2};$$

thus, using (4.84),

$$u_{xx} = \frac{1}{r}f'(r) + \frac{x^2}{r^2}f''(r) - \frac{x^2}{r^3}f'(r), \quad \text{for } r > 0. \tag{4.88}$$

Similar calculations, using (4.85) and (4.87) yield

$$u_{yy} = \frac{1}{r}f'(r) + \frac{y^2}{r^2}f''(r) - \frac{y^2}{r^3}f'(r), \quad \text{for } r > 0. \tag{4.89}$$

Next, add the expressions in (4.88) and (4.89) to obtain

$$u_{xx} + u_{yy} = \frac{2}{r}f'(r) + \frac{x^2 + y^2}{r^2}f''(r) - \frac{x^2 + y^2}{r^3}f'(r), \quad \text{for } r > 0,$$

or using the fact that $x^2 + y^2 = r^2$,

$$u_{xx} + u_{yy} = \frac{2}{r}f'(r) + f''(r) - \frac{1}{r}f'(r), \quad \text{for } r > 0,$$

or

$$u_{xx} + u_{yy} = f''(r) + \frac{1}{r}f'(r), \quad \text{for } r > 0. \tag{4.90}$$

It follows from (4.90) that, if u in (4.82) solves Laplace's equation in R, then f solves the second order ODE

$$f''(r) + \frac{1}{r}f'(r) = 0, \quad \text{for } r > 0,$$

or

$$rf''(r) + f'(r) = 0, \quad \text{for } r > 0,$$

which can be rewritten as

$$\frac{d}{dr}[rf'(r)] = 0, \quad \text{for } r > 0. \tag{4.91}$$

Integrating the equation in (4.91) yields

$$rf'(r) = c_1, \quad \text{for } r > 0,$$

and some constant c_1, or

$$f'(r) = \frac{c_1}{r}, \quad \text{for } r > 0, \tag{4.92}$$

and some constant c_1. Integrating the equation in (4.92) yields

$$f(r) = c_1 \ln r + c_2, \quad \text{for } r > 0, \tag{4.93}$$

and constants c_1 and c_2.

It follows from (4.82) and (4.93) that radially symmetric solutions of (4.81) are of the form

$$u(x, y) = c_1 \ln \sqrt{x^2 + y^2} + c_2, \quad \text{for } (x, y) \neq (0, 0), \tag{4.94}$$

and constants c_1 and c_2. □

Example 4.2.2 (The Dirichlet Problem in an Annulus)**.** For positive numbers, r_1 and r_2, with $r_1 < r_2$, define R to be the annulus

$$R = \{(x, y) \in \mathbb{R}^2 \mid r_1 < \sqrt{x^2 + y^2} < r_2\}. \tag{4.95}$$

Denote by C_r the circle of radius r centered at the origin.

Solve the boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } R; \\ u = a, & \text{on } C_{r_1}; \\ u = b, & \text{on } C_{r_2}, \end{cases} \tag{4.96}$$

where a and b are real constants.

Solution: Since the annulus R in (4.95) has radial symmetry, and the boundary conditions in (4.96) are also radially symmetric, it makes sense to look for radially symmetric solutions of problem (4.96). According to the result of Example 4.2.1, these are of the form given in (4.94); namely

$$u(x,y) = c_1 \ln\sqrt{x^2+y^2} + c_2, \quad \text{for } (x,y) \in R, \tag{4.97}$$

for some constants c_1 and c_2.

The boundary conditions In (4.96) then imply that

$$c_1 \ln r_1 + c_2 = a \tag{4.98}$$

and

$$c_1 \ln r_2 + c_2 = b, \tag{4.99}$$

in view of (4.97). Solving the system of equations in (4.98) and (4.99) for c_1 and c_2 yields

$$c_1 = \frac{b-a}{\ln(r_2/r_1)},$$

and

$$c_2 = \frac{a\ln r_2 - b\ln r_1}{\ln(r_2/r_1)}.$$

Substituting these values for c_1 and c_2 into (4.97) yields a solution,

$$u(x,y) = \frac{b-a}{\ln(r_2/r_1)} \ln\sqrt{x^2+y^2} + \frac{a\ln r_2 - b\ln r_1}{\ln(r_2/r_1)}, \quad \text{for } (x,y) \in R, \tag{4.100}$$

to the BVP in (4.96). The result of Problem 3 in Assignment #7 then shows that the function u given in (4.100) is the solution of the BVP in (4.96). □

4.2.2 Dilation Invariant Solutions to Laplace's Equation

In this section we explore the effect of the change of variables

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \tag{4.101}$$

for nonzero real constants α and β, on the two–dimensional Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad \text{in } \mathbb{R}^2. \tag{4.102}$$

The change of variables in (4.101) corresponds to

$$\begin{cases} \xi = \alpha x; \\ \eta = \beta y, \end{cases} \tag{4.103}$$

or

$$\begin{cases} x = \xi/\alpha; \\ y = \eta/\beta. \end{cases} \tag{4.104}$$

Setting

$$v(\xi, \eta) = u(x, y), \tag{4.105}$$

where x and y are given in terms of ξ and η by the equations in (4.104), we compute, using the Chain Rule, we obtain from (4.105) that

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x},$$

where, by virtue of the equations in (4.103),

$$\frac{\partial \xi}{\partial x} = \alpha \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0,$$

so that

$$u_x = \alpha v_\xi. \tag{4.106}$$

Similarly,

$$u_y = \beta v_\eta. \tag{4.107}$$

Next, differentiate on both sides of (4.106) and apply the Chain Rules as in the previous calculations to obtain

$$u_{xx} = \alpha^2 v_{\xi\xi}. \tag{4.108}$$

Similarly, we obtain from (4.107) that

$$u_{yy} = \beta^2 v_{\eta\eta}. \tag{4.109}$$

Adding (4.108) and (4.109) we obtain

$$u_{xx} u_{yy} = \alpha^2 v_{\xi\xi} + \beta^2 v_{\eta\eta}. \tag{4.110}$$

Thus, if u solves Laplace's equation in (4.102), we obtain from (4.110) that

$$\alpha^2 v_{\xi\xi} + \beta^2 v_{\eta\eta} = 0. \tag{4.111}$$

It follows from (4.111) that Laplace's equation in $\mathbb{R}^2$ is invariant under the scaling transformation in (4.101), provided that $\alpha^2 = \beta^2$. We will therefore set $\alpha = \beta = \lambda$ in (4.101) to get

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = D_\lambda \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.112}$$

where D_λ denotes the scalar matrix

$$D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

for a nonzero parameter λ. The transformations in (4.112) form a one–parameter family of dilations corresponding to the change of variables

$$\left\{ \begin{array}{rcl} \xi & = & \lambda x; \\ \eta & = & \lambda y. \end{array} \right. \tag{4.113}$$

Note from (4.113) that a combination of the variables that is independent of the dilation parameter λ is

$$\frac{\eta}{\xi} = \frac{y}{x}, \qquad \text{for } x \neq 0.$$

This suggests that we look for solutions to Laplace's equation in $\mathbb{R}^2$ of the form

$$u(x,y) = f\left(\frac{y}{x}\right), \qquad \text{for } x \neq 0, \tag{4.114}$$

where f is a C^2 function of a single variable.

Set

$$s = \frac{y}{x}, \qquad \text{for } x \neq 0, \tag{4.115}$$

so that, in view of (4.114)

$$u(x,y) = f(s), \tag{4.116}$$

where s is given by (4.115).

We look for a solution to Laplace's equation in $\mathbb{R}^2$ of the form in (4.116) where f is a C^2 function and s is as given in (4.115). Thus, assume that u solves (4.102) and compute

$$u_x = f'(s)\frac{\partial s}{\partial x}, \tag{4.117}$$

where we have used the Chain Rule and where

$$\frac{\partial s}{\partial x} = -\frac{y}{x^2}, \qquad \text{for } x \neq 0,$$

by virtue of (4.115), so that

$$\frac{\partial s}{\partial x} = -\frac{s}{x}, \qquad \text{for } x \neq 0. \tag{4.118}$$

Substituting (4.118) into the right–hand side of (4.117) then yields

$$u_x = -\frac{1}{x}sf'(s), \qquad \text{for } x \neq 0. \tag{4.119}$$

Next, differentiate with respect to x on both sides of (4.119) to get

$$u_{xx} = \frac{1}{x^2}sf'(s) - \frac{1}{x}\frac{\partial s}{\partial x}f'(s) - \frac{1}{x}sf''(s)\frac{\partial s}{\partial x}, \qquad \text{for } x \neq 0, \tag{4.120}$$

where we have used the Product Rule and the Chain Rule. Then, substitute (4.118) into the right–hand side of (4.120) to get

$$u_{xx} = \frac{2s}{x^2}f'(s) + \frac{s^2}{x^2}f''(s), \qquad \text{for } x \neq 0. \tag{4.121}$$

Next, apply the Chain Rule to obtain from (4.116) that

$$u_y = f'(s)\frac{\partial s}{\partial y}, \tag{4.122}$$

where

$$\frac{\partial s}{\partial y} = \frac{1}{x}, \qquad \text{for } x \neq 0. \tag{4.123}$$

It then follows from (4.122) and (4.123) that

$$u_y = \frac{1}{x} f'(s), \qquad \text{for } x \neq 0. \tag{4.124}$$

Differentiate on both sides of (4.124) with respect to y, apply the Chain Rule, and use (4.123) to obtain

$$u_{yy} = \frac{1}{x^2} f''(s), \qquad \text{for } x \neq 0. \tag{4.125}$$

Next, add the expressions in (4.121) and (4.125) to get

$$u_{xx} + u_{yy} = \frac{2s}{x^2} f'(s) + \frac{1+s^2}{x^2} f''(s), \qquad \text{for } x \neq 0. \tag{4.126}$$

It follows from (4.126) that, if u solves Laplace's equation in $\mathbb{R}^2$, then f solves the second order ODE

$$\frac{2s}{x^2} f'(s) + \frac{1+s^2}{x^2} f''(s) = 0, \qquad \text{for } x \neq 0,$$

or

$$(1+s^2) f''(s) + 2s f'(s) = 0. \tag{4.127}$$

In order to solve the ODE in (4.127), set

$$v(s) = f'(s), \tag{4.128}$$

so that

$$(1+s^2)\frac{dv}{ds} + 2sv = 0. \tag{4.129}$$

The first order ODE in (4.129) can be solved by separating variables to yield

$$\int \frac{1}{v}\, dv = -\int \frac{2s}{1+s^2}\, ds,$$

or

$$\ln |v| = \ln\left(\frac{1}{1+s^2}\right) + c_o, \tag{4.130}$$

for some constant c_o.

Exponentiating on both sides of (4.130) and using the continuity of v we obtain

$$v(s) = \frac{c_1}{1+s^2}, \qquad \text{for } s \in \mathbb{R}, \tag{4.131}$$

and some constant c_1. It follows from (4.128) and (4.131) that

$$f'(s) = \frac{c_1}{1+s^2}, \qquad \text{for } s \in \mathbb{R},$$

and some constant c_1, which can be integrated to yield

$$f(s) = c_1 \arctan(s) + c_2, \quad \text{for } s \in \mathbb{R}, \tag{4.132}$$

and constants c_1 and c_2.

It follows from (4.114) and (4.132) that dilation–invariant solutions of Laplace's equation in $\mathbb{R}^2$ are of the form

$$u(x, y) = c_1 \arctan\left(\frac{y}{x}\right) + c_2, \quad \text{for } x \neq 0, \tag{4.133}$$

and constants c_1 and c_2. The result in (4.133) states that dilation–invariant harmonic functions in $\mathbb{R}^2$ are linear functions of the angle, θ, the the point (x, y), for $(x, y) \neq (0, 0)$, makes with the positive x–axis:

$$u = c_1\theta + c_2,$$

for constants c_1 and c_2.

4.2.3 Dilation Invariant Solutions of the Diffusion Equation

In this section we look for dilation–invariant solutions of the one–dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4.134}$$

where $D > 0$ is the diffusivity constant. We proceed as in Section 4.2.2 by finding conditions on parameters α and β so that the diffusion equation in (4.134) is invariant under the change of variables

$$\begin{cases} \xi = \alpha x; \\ \tau = \beta t, \end{cases} \tag{4.135}$$

where $\alpha\beta \neq 0$.

Write

$$v(\xi, \tau) = u(x, t), \tag{4.136}$$

where x and t are given in terms of ξ and τ by inverting the system in (4.136),

$$\begin{cases} x = \xi/\alpha; \\ t = \tau/\beta, \end{cases}$$

and use the Chain Rule to compute

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\tau \frac{\partial \tau}{\partial x},$$

$$\frac{\partial \xi}{\partial x} = \alpha \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0,$$

so that

$$u_x = \alpha v_\xi. \tag{4.137}$$

Similarly,

$$u_t = \beta v_\tau. \tag{4.138}$$

Next, differentiate on both sides of (4.137) and apply the Chain Rules as in the previous calculations to obtain

$$u_{xx} = \alpha^2 v_{\xi\xi}. \tag{4.139}$$

Using the expressions in (4.138) and (4.139) we obtain

$$u_t - D\ u_{xx} = \beta v_\tau - D\alpha^2 v_{\xi\xi},$$

so that, if u solves the diffusion equation in (4.134),

$$\beta v_\tau - D\alpha^2 v_{\xi\xi} = 0. \tag{4.140}$$

Hence, the diffusion equation in (4.134) is invariant under the change of variables in (4.136) provided that

$$\beta = \alpha^2. \tag{4.141}$$

It follows from (4.140) and (4.141) that the diffusion equation in (4.134) is invariant under the dilation

$$\begin{cases} \xi &= \alpha x; \\ \tau &= \alpha^2 t. \end{cases} \tag{4.142}$$

It follows from (4.142) that combinations of the variables that are independent of the dilation parameter, α, are

$$\frac{\xi^2}{\tau} = \frac{x^2}{t} \quad \text{or} \quad \frac{\xi}{\sqrt{\tau}} = \frac{x}{\sqrt{t}}, \quad \text{for } \tau > 0 \text{ and } t > 0.$$

Thus, in order to find dilation–invariant solutions of the one–dimensional diffusion equation, we look for solutions of the form

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right), \quad \text{for } t > 0, \tag{4.143}$$

where f is a C^2 function of a single variable.

Set

$$s = \frac{x}{\sqrt{t}}, \quad \text{for } t > 0, \tag{4.144}$$

so that, in view of (4.143)

$$u(x,t) = f(s), \tag{4.145}$$

where s is given by (4.144).

Differentiate on both sides of (4.145) with respect to x, using the Chain Rule, to get

$$u_x = f'(s)\frac{\partial s}{\partial x},$$

where

$$\frac{\partial s}{\partial x} = \frac{1}{\sqrt{t}}, \quad \text{for } t > 0, \tag{4.146}$$

by virtue of (4.144), so that

$$u_x = \frac{1}{\sqrt{t}} f'(s), \quad \text{for } t > 0. \tag{4.147}$$

Differentiate with respect to x on both sides of (4.147), use the Chain Rule, and the result in (4.146) to get

$$u_{xx} = \frac{1}{t} f''(s), \quad \text{for } t > 0. \tag{4.148}$$

Next, differentiate on both sides of (4.145) with respect to t, using the Chain Rule, to get

$$u_t = f'(s)\frac{\partial s}{\partial t}, \tag{4.149}$$

where, by virtue of (4.144),

$$\frac{\partial s}{\partial t} = -\frac{x}{2t\sqrt{t}},$$

or, using (4.144),

$$\frac{\partial s}{\partial t} = -\frac{s}{2t}, \quad \text{for } t > 0. \tag{4.150}$$

Substitute the result in (4.150) into the right–hand side of (4.149) to get

$$u_t = -\frac{s}{2t} f'(s), \quad \text{for } t > 0. \tag{4.151}$$

It follows from (4.148) and (4.151) that, if u given in (4.145) solves the diffusion equation in (4.134), then f solves the ODE

$$-\frac{s}{2t} f'(s) = \frac{D}{t} f''(s), \quad \text{for } t > 0,$$

or

$$f''(s) + \frac{s}{2D} f'(s) = 0 \tag{4.152}$$

In order to solve the ODE in (4.152), set

$$v(s) = f'(s), \tag{4.153}$$

so that

$$\frac{dv}{ds} + \frac{s}{2D} v = 0. \tag{4.154}$$

The first order ODE in (4.154) can be solved by separating variables to yield

$$\int \frac{1}{v}\, dv = -\int \frac{s}{2D}\, ds,$$

or

$$\ln|v| = -\frac{s^2}{4D} + c_o, \tag{4.155}$$

for some constant c_o.

Exponentiating on both sides of (4.155) and using the continuity of v we obtain

$$v(s) = c_1 e^{-s^2/4D}, \quad \text{for } s \in \mathbb{R}, \tag{4.156}$$

and some constant c_1. It follows from (4.156) and (4.153) that

$$f'(s) = c_1 e^{-s^2/4D}, \quad \text{for } s \in \mathbb{R},$$

and some constant c_1, which can be integrated to yield

$$f(s) = c_1 \int_0^s e^{-z^2/4D}\, dz + c_2, \quad \text{for } s \in \mathbb{R}, \tag{4.157}$$

and constants c_1 and c_2. It follows from (4.143) and (4.157) that dilation–invariant solutions of one–dimensional diffusion equation in (4.134) are of the form

$$u(x,t) = c_1 \int_0^{x/\sqrt{t}} e^{-z^2/4D}\, dz + c_2, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4.158}$$

and constants c_1 and c_2.

Chapter 5

Solving Linear PDEs

In Chapter 4 we saw two general approaches for finding solutions to first or second order PDEs: using characteristic curves and looking for symmetric solutions. In theory, these methods could be applied to nonlinear or linear equations. In this chapter we explore methods that exploit the special structure provided by linear PEDs. In Section 3.1 we saw the Principle of Superposition (Proposition 3.1.1 on page 42 in these notes), which states that linear combinations of solutions to the homogeneous linear PDE

$$Lu = 0,$$

where L is a linear differential operator, are also solutions. Thus, in principle, we can use superposition to construct solutions of linear PDEs satisfying certain conditions by putting together known solutions. We will see in this chapter that this procedure can be carried out by adding together infinitely many solutions in the form of a series or and integral transform. We will begin by presenting some special solutions that can be used as building blocks to obtain solutions to initial and/or boundary value problems for a large class of linear PDEs. These special solutions are known as **Fundamental Solutions**.

5.1 Fundamental Solutions

We will illustrate the concept of a fundamental solution by first finding a special solution to the one–dimensional diffusion equation.

5.1.1 Fundamental Solution to the Diffusion Equation

We compute a very special solution to the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.1}$$

In Section 4.2.3 we derived the following dilation–invariant solution to the diffusion equation in (5.1):

$$u(x,t) = c_1 \int_0^{x/\sqrt{t}} e^{-z^2/4D}\ dz + c_2, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.2}$$

and constants c_1 and c_2. Observe that the function $u\colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$ defined in (5.2) is a composition of C^∞ functions. It then follows by the Fundamental Theorem of Calculus and the Chain Rule that $u \in C^\infty(\mathbb{R} \times (0,\infty))$. Hence, we can differentiate on both sides of the PDE in (5.1) with respect to x, for example, and get the valid statement

$$u_{tx} = D\ u_{xxx};$$

thus, by the equality of the mixed partial derivatives,

$$(u_x)_t = D\ (u_x)_{xx},$$

which shows that u_x is also a solution of the one–dimensional diffusion equation in (5.1). Hence, by taking the partial derivative with respect to x in (5.2) we obtain another solution to the the one–dimensional diffusion equation in (5.1). Set $v(x,t) = u_x(x,t)$, for $x \in \mathbb{R}$ and $t > 0$, where u is given in (5.2). Then, using the Fundamental Theorem of Calculus and the Chain Rule, we obtain from (5.2) that

$$v(x,t) = \frac{c_1}{\sqrt{t}}\ e^{-x^2/4Dt}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.3}$$

and some constant c_1, is a solution to the one–dimensional diffusion equation in (5.1).

An interesting property of the function defined in (5.3) is that the integral $\displaystyle\int_{-\infty}^{\infty} v(x,t)\ dx$ is finite and is independent of $t > 0$. Indeed, using the fact that

$$\int_{-\infty}^{\infty} e^{-z^2}\ dz = \sqrt{\pi},$$

and making the change of variables

$$z = \frac{x}{\sqrt{4Dt}},$$

so that

$$dx = \sqrt{4Dt}\ dz,$$

we obtain, for $t > 0$,

$$\int_{-\infty}^{\infty} v(x,t)\ dx = \frac{c_1}{\sqrt{t}}\sqrt{4Dt}\int_{-\infty}^{\infty} e^{-z^2}\ dz,$$

or

$$\int_{-\infty}^{\infty} v(x,t)\ dx = c_1\sqrt{4D\pi}, \qquad \text{for all } t > 0. \tag{5.4}$$

We chose the constant c_1 in (5.4) so that

$$\int_{-\infty}^{\infty} v(x,t)\ dx = 1, \qquad \text{for all } t > 0;$$

that is,

$$c_1 = \frac{1}{\sqrt{4D\pi}}. \tag{5.5}$$

Substituting the value of c_1 in (5.5) into the definition of $v(x,t)$ in (5.3), we obtain

$$v(x,t) = \frac{1}{\sqrt{4\pi Dt}}\ e^{-x^2/4Dt}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.6}$$

We shall denote the expression for $v(x,t)$ defined in (5.6) by $p(x,t)$, so that

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\ e^{-x^2/4Dt}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.7}$$

It then follows from what we have shown thus far that the function p defined in (5.7) is a C^∞ function defined in $\mathbb{R} \times (0,\infty)$ that solves the one–dimensional diffusion equation in (5.1); that is,

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.8}$$

Also, it follows from (5.4) and (5.5) that

$$\int_{-\infty}^{\infty} p(x,t)\ dx = 1, \qquad \text{for all } t > 0. \tag{5.9}$$

In fact, using a change of variables we obtain from (5.9) that

$$\int_{-\infty}^{\infty} p(x-y,t)\ dy = 1, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0. \tag{5.10}$$

In addition to (5.10), the function p defined in (5.7) has the following properties:

Proposition 5.1.1 (Properties of p)**.** Let $p(x,t)$ be as defined in (5.7) for $x \in \mathbb{R}$ and $t > 0$.

(i) $p(x-y,t) > 0$ for all $x, y \in \mathbb{R}$ and $t > 0$

(ii) If $x \neq y$, then $\lim_{t\to 0^+} p(x-y,t) = 0$.

(iii) If $x = y$, then $\lim_{t\to 0^+} p(x-y,t) = +\infty$.

See Problem 5 in Assignment #14.

In this section we will see how to use the properties in (5.10) and in Proposition 5.1.1 to obtain a solution to the initial value problem for the one–dimensional diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{5.11}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded function that is also **piecewise continuous**.

Definition 5.1.2 (Piecewise Continuous Functions)**.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have a **jump discontinuity** at $x \in \mathbb{R}$ if the one–sided limits

$$\lim_{y\to x^+} f(y) \quad \text{and} \quad \lim_{y\to x^-} f(y)$$

exist and

$$\lim_{y\to x^+} f(y) \neq \lim_{y\to x^-} f(y).$$

We say that f is piecewise continuous if it is continuous except at an at most countable number of points at which f has jump discontinuities.

Figure 5.1.1 shows a portion of the sketch of a piecewise continuous function. We will show that the function $u\colon \mathbb{R}\times(0,\infty) \to \mathbb{R}$ given by

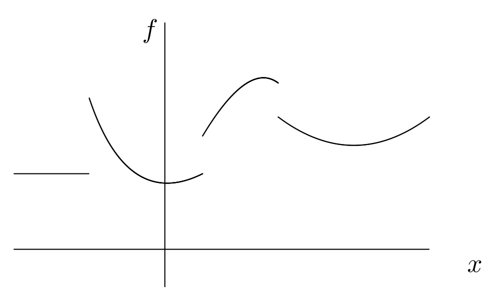

Figure 5.1.1: Sketch of a Piecewise Continuous Function

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)f(y)\ dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.12}$$

is a candidate for a solution of the initial value problem in (5.11). We note that, since $p(x-y,t))$ is not defined at $t = 0$, the initial condition in the IVP in (5.11) has to be understood as

$$\lim_{t\to 0^+} u(x,t) = f(x).$$

We will see in this section that (5.13) holds true foe values of x at which f is continuous. For values of x at which f has a jump discontinuity

$$\lim_{t\to 0^+} u(x,t) = \frac{f(x^+) + f(x^-)}{2},$$

where $f(x^+)$ and $f(x^-)$ are the one–sided limits

$$f(x^+) = \lim_{y\to x^+} f(y) \quad \text{and} \quad f(x^-) = \lim_{y\to x^-} f(y),$$

respectively.

We state the main result of this section as the following proposition:

Proposition 5.1.3. Let u be given by (5.12), where $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded, piecewise continuous function. Then, u is $C^{2,1}(\mathbb{R}\times(0,\infty))$[1] and

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t), \quad \text{for } x\in\mathbb{R} \text{ and } t>0. \tag{5.13}$$

Furthermore,

$$\lim_{t\to 0^+} u(x,t) = f(x). \tag{5.14}$$

if f is continuous at x, and

$$\lim_{t\to 0^+} u(x,t) = \frac{f(x^+) + f(x^-)}{2}, \tag{5.15}$$

if f has a jump discontinuity at x.

Once we have proved Proposition 5.1.3, we will have constructed a solution

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)f(y)\ dy, \quad \text{for } x\in\mathbb{R} \text{ and } t>0, \tag{5.16}$$

to the initial value problem of the initial value for the one–dimensional diffusion for the case of continuous initial data f, where p is defined in (5.7). Thus, a solution of the initial value problem in (5.11) is obtained by integrating $f(y)p(x-y,t)$ over y in the entire real line. The map

$$(x,y,t)\mapsto p(x-y,t), \quad \text{for all } x,y\in\mathbb{R} \text{ and } t>0,$$

or

$$(x,y,t)\mapsto \frac{1}{\sqrt{4\pi Dt}}\ e^{-(x-y)^2/4Dt}, \quad \text{for all } x,y\in\mathbb{R} \text{ and } t>0,$$

is usually called the **heat kernel**; we shall also call it the **fundamental solution** to the one–dimensional diffusion equation. We will denote it by $K(x,y,t)$, so that $K\colon \mathbb{R}^2\times(0,\infty)\to\mathbb{R}$ and

$$K(x,y,t) = \frac{1}{\sqrt{4\pi Dt}}\ e^{-(x-y)^2/4Dt}, \quad \text{for all } x,y\in\mathbb{R} \text{ and } t>0. \tag{5.17}$$

[1]The function u is C^2 in the first variable, and C^1 in the second variable

We shall reiterate the properties of the heat kernel that we have discussed for future reference in the following proposition, we will add the additional observation that K is symmetric in x and y; that is $K(x, y, t) = K(y, x, t)$ for all $x, y \in \mathbb{R}$ and $t > 0$.

Proposition 5.1.4 (Properties of the Heat Kernel)**.** Let $K(x, y, t)$ be as defined in (5.17) for $x, y \in \mathbb{R}$ and $t > 0$.

(i) $K(x, y, t) = K(y, x, t)$ for all $x, y \in \mathbb{R}$ and $t > 0$.

(ii) $K(x, y, t) > 0$ for all $x, y \in \mathbb{R}$ and $t > 0$.

(iii) $\displaystyle\int_{-\infty}^{\infty} K(y, x, t)\ dy = 1$ for all $x \in \mathbb{R}$ and $t > 0$.

(iv) If $x \neq y$, then $\displaystyle\lim_{t\to 0^+} K(x, y, t) = 0$.

(v) If $x = y$, then $\displaystyle\lim_{t\to 0^+} K(x, y, t) = +\infty$.

Before we prove Proposition 5.1.3, we will establish two Lemmas; the first one involves the error function,

$$\text{Erf}\colon \mathbb{R} \to \mathbb{R},$$

defined by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2}\ dr, \qquad \text{for } x \in \mathbb{R}, \tag{5.18}$$

and its properties:

Proposition 5.1.5. Let $\text{Erf}\colon \mathbb{R} \to \mathbb{R}$ be as given in (5.18). Then,

(i) $\text{Erf}(0) = 0$;

(ii) $\displaystyle\lim_{x\to\infty} \text{Erf}(x) = 1$;

(iii) $\displaystyle\lim_{x\to-\infty} \text{Erf}(x) = -1$;

See Problem 1 in Assignment #14.

A sketch of the graph of $y = \text{Erf}(x)$ is shown in Figure 5.1.2.

Lemma 5.1.6. Let $p(x, t)$ be as defined in (5.7) for $x \in \mathbb{R}$ and $t > 0$. For $\delta > 0$,

$$\lim_{t\to 0^+} \int_\delta^\infty p(x, t)\ dx = 0. \tag{5.19}$$

and

$$\lim_{t\to 0^+} \int_{-\infty}^{-\delta} p(x, t)\ dx = 0. \tag{5.20}$$

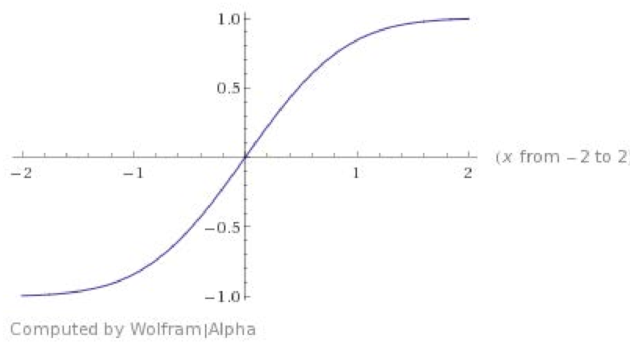

Figure 5.1.2: Sketch of Graph of Error Function

Proof: Make the change of variables $y = \dfrac{x}{\sqrt{4Dt}}$ to write

$$\begin{aligned}\int_\delta^\infty p(x,t)\ dx &= \int_\delta^\infty \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-x^2/4Dt}\ dx \\ &= \frac{1}{\sqrt{\pi}} \int_{\delta/\sqrt{4Dt}}^\infty e^{-y^2}\ dy \\ &= \frac{1}{2}\left[1 - \mathrm{Erf}\left(\frac{\delta}{\sqrt{4Dt}}\right)\right],\end{aligned}$$

where we have used the definition of the error function in (5.18) and the fact that

$$\int_0^\infty e^{-y^2}\ dy = \frac{\sqrt{\pi}}{2}.$$

We then have that

$$\int_\delta^\infty p(x,t)\ dx = \frac{1}{2}\left[1 - \mathrm{Erf}\left(\frac{\delta}{\sqrt{4Dt}}\right)\right], \quad \text{for } t > 0. \tag{5.21}$$

Now, it follows from (5.21) and (ii) in Proposition 5.1.5 that

$$\lim_{t\to 0^+} \int_\delta^\infty p(x,t)\ dx = 0,$$

which is (5.19). Similar calculations can be used to derive (5.20). ■

Lemma 5.1.7. Let $p(x,t)$ be as defined in (5.7) for $x \in \mathbb{R}$ and $t > 0$. Then, we have the following estimates on integrals of the absolute values of the derivatives of p:

$$\int_{-\infty}^\infty \left|\frac{\partial p}{\partial t}(x-y,t)\right|\ dy \leqslant \frac{1}{t}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0, \tag{5.22}$$

and

$$\int_{-\infty}^{\infty} \left| \frac{\partial p}{\partial x}(x-y,t) \right| \, dy = \frac{1}{\sqrt{\pi D t}}, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0, \tag{5.23}$$

Proof: Compute the partial derivative of

$$p(x-y,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-(x-y)^2/4Dt}, \qquad \text{for all } x, y \in \mathbb{R} \text{ and } t > 0, \tag{5.24}$$

with respect to t to obtain

$$\frac{\partial}{\partial t}[p(x-y,t)] = -\frac{1}{2t} p(x-y,t) + \frac{(x-y)^2}{4Dt^2} p(x-y,t), \tag{5.25}$$

for all $x, y \in \mathbb{R}$ and $t > 0$. Next, take absolute value on both sides of (5.25), apply the triangle inequality, and use the positivity of the heat kernel (see (ii) in Proposition 5.1.4) to get

$$\left| \frac{\partial}{\partial t}[p(x-y,t)] \right| \leqslant \frac{1}{2t} p(x-y,t) + \frac{(x-y)^2}{4Dt^2} p(x-y,t), \tag{5.26}$$

for all $x, y \in \mathbb{R}$ and $t > 0$. Integrating on both sides of (5.26) and using (5.10) (see (iii) in Proposition 5.1.4) yields

$$\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t}[p(x-y,t)] \right| \, dy \leqslant \frac{1}{2t} + \int_{-\infty}^{\infty} \frac{(x-y)^2}{4Dt^2} p(x-y,t)\, dy, \tag{5.27}$$

for all $x \in \mathbb{R}$ and $t > 0$.

Next, we evaluate the right–most integral in (5.27),

$$\int_{-\infty}^{\infty} \frac{(x-y)^2}{4Dt^2} p(x-y,t)\, dy = \int_{-\infty}^{\infty} \frac{(x-y)^2}{4Dt^2} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi D t}}\, dy,$$

by making the change of variables

$$\xi = \frac{y-x}{\sqrt{4Dt}},$$

so that

$$\int_{-\infty}^{\infty} \frac{(x-y)^2}{4Dt^2} p(x-y,t)\, dy = \frac{1}{t\sqrt{\pi}} \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2}\, d\xi, \tag{5.28}$$

for all $x \in \mathbb{R}$ and $t > 0$. The right–most integral in (5.28) can be evaluated using integration by parts to yield

$$\begin{aligned} \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2}\, d\xi &= 2\int_{0}^{\infty} \xi^2 e^{-\xi^2}\, d\xi \\ &= -\xi e^{-\xi^2}\Big|_0^{\infty} + \int_0^{\infty} e^{-\xi^2}\, d\xi, \end{aligned}$$

so that

$$\int_{-\infty}^{\infty} \xi^2 e^{-\xi^2}\ d\xi = \frac{\sqrt{\pi}}{2}. \tag{5.29}$$

Combining (5.29), (5.28) and (5.27) yields the estimate

$$\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t}[p(x-y,t)] \right|\ dy \leqslant \frac{1}{t}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

which is (5.22).

In order to establish (5.23), first take the partial derivative with respect to x on both side of (5.24) to get

$$\frac{\partial}{\partial x}[p(x-y,t)] = -\frac{x-y}{2Dt} p(x-y,t), \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.30}$$

so that, taking absolute value on both sides of (5.30) and integrating,

$$\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x}[p(x-y,t)] \right|\ dy = \int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} p(x-y,t)\ dy, \tag{5.31}$$

Evaluate the right–most integral in (5.31),

$$\int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} p(x-y,t)\ dy = \int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}}\ dy, \tag{5.32}$$

by making the change of variables

$$\xi = \frac{y-x}{\sqrt{4Dt}},$$

to get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}}\ dy &= \frac{1}{\sqrt{\pi Dt}} \int_{-\infty}^{\infty} |\xi| e^{-\xi^2}\ d\xi \\ &= \frac{2}{\sqrt{\pi Dt}} \int_{0}^{\infty} \xi e^{-\xi^2}\ d\xi, \end{aligned}$$

so that

$$\int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}}\ dy = \frac{1}{\sqrt{\pi Dt}}. \tag{5.33}$$

The statement in (5.23) now follows by putting together the results in (5.33), (5.32) and (5.31). ■

Proof of Proposition 5.1.3: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function satisfying

$$|f(x)| \leqslant M, \quad \text{for all } x \in \mathbb{R}, \tag{5.34}$$

and some positive constant M, and define $u\colon \mathbb{R} \times (0,t) \to \mathbb{R}$ by

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)f(y)\ dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.35}$$

where $p(x-y,t)$ denotes the heat kernel given in (5.24). We will show that u solves the one–dimensional diffusion equation in (5.13). Before we do that, though, we need to verify that the expression in (5.35) does indeed define a function $u\colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$. In order to do this we need to make sure that the integral on the right–hand side of (5.35) is a real number. This will follow from the estimate

$$\int_{-\infty}^{\infty} |p(x-y,t)f(y)|\ dy < \infty \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.36}$$

In order to derive the estimate in (5.36), use the positivity of the heat kernel (see (ii) in Proposition 5.1.4), (5.10) and (5.34) to compute

$$\int_{-\infty}^{\infty} |p(x-y,t)f(y)|\ dy \leqslant M \int_{-\infty}^{\infty} p(x-y,t)\ dy,$$

so that

$$\int_{-\infty}^{\infty} |p(x-y,t)f(y)|\ dy \leqslant M, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.37}$$

which implies (5.36). Observe that the estimate in (5.37) also implies that

$$|u(x,t)| \leqslant M, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

by virtue of (5.35).

The fact that u defined in (5.35) solves the one–dimensional diffusion equation in (5.13) will follow from the fact that the heat kernel itself solves the one–dimensional heat equation,

$$\frac{\partial}{\partial t}[p(x-y,t)] = D\frac{\partial^2}{\partial x^2}[p(x-y,t)], \quad \text{for } x, y \in \mathbb{R} \text{ and } t > 0; \tag{5.38}$$

(see also (5.8). Indeed, suppose for the moment that we can interchange differentiation and integration in the definition of u in (5.35), so that

$$\frac{\partial u}{\partial t}(x,t) = \int_{-\infty}^{\infty} \frac{\partial p}{\partial t}(x-y,t)f(y)\ dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.39}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \int_{-\infty}^{\infty} \frac{\partial^2 p}{\partial x^2}(x-y,t)f(y)\ dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.40}$$

Thus, combining (5.39) and (5.40),

$$\frac{\partial u}{\partial t}(x,t) - D\frac{\partial^2 u}{\partial x^2}(x,t) = \int_{-\infty}^{\infty} \left[\frac{\partial p}{\partial t}(x-y,t) - D\frac{\partial^2 p}{\partial x^2}(x-y,t)\right] f(y)\ dy,$$

which shows that (5.13) holds true by virtue of (5.38)

The expressions in (5.39) and (5.40) are justified by the assumption that f is bounded (see (5.34) and the estimates (5.22) and (5.23) in Lemma 5.1.7; namely,

$$\int_{-\infty}^{\infty} \left|\frac{\partial p}{\partial t}(x-y,t)\right|\ dy \leqslant \frac{1}{t}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

and

$$\int_{-\infty}^{\infty} \left|\frac{\partial p}{\partial x}(x-y,t)\right|\ dy = \frac{1}{\sqrt{\pi D t}}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

Observe that, (5.40) and (5.38) imply the estimate

$$\int_{-\infty}^{\infty} \left|\frac{\partial^2 p}{\partial x^2}(x-y,t)\right|\ dy \leqslant \frac{1}{Dt}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

We have therefore established that the function $u\colon \mathbb{R}\times(0,1)$ defined in (5.35) is a C^2 function in the first variable, C^1 in the second variable, and is a solution to the one–dimensional diffusion equation.

Next, we will prove the second assertion in Proposition 5.1.3.

(i) Assume first that f is continuous at x and let $\varepsilon > 0$ be given. Then, there exists $\delta > 0$ such that

$$|y-x| < \delta \Rightarrow |f(y)-f(x)| < \frac{\varepsilon}{3}. \tag{5.41}$$

We consider

$$u(x,t) - f(x) = \int_{-\infty}^{\infty} p(x_o - y,t)f(y)\ dy - f(x)\int_{-\infty}^{\infty} p(x_o - y,t)\ dy,$$

where we have used the definition of $u(x,t)$ in (5.35) and (5.10) (see also the fact (iii) in Proposition 5.1.4). We then have that

$$u(x,t) - f(x) = \int_{-\infty}^{\infty} p(x-y,t)(f(y)-f(x))\ dy,$$

so that

$$|u(x,t) - f(x)| \leqslant \int_{-\infty}^{\infty} p(x-y,t)|f(y)-f(x)|\ dy, \tag{5.42}$$

where we have used the fact that $p(x,t)$ is positive for all $x \in \mathbb{R}$ and all $t > 0$.

Next, re-write the integral on the right–hand side of (5.42) as a sum of three integrals,

$$\begin{aligned}\int_{-\infty}^{\infty} p(x-y, \quad t)|f(y)-f(x)|\, dy = \\ &\int_{-\infty}^{x-\delta} p(x-y,t)|f(y)-f(x)|\, dy \\ &+\int_{x-\delta}^{x+\delta} p(x-y,t)|f(y)-f(x)|\, dy \\ &+\int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\, dy.\end{aligned} \tag{5.43}$$

We first estimate the middle integral on the right–hand side of (5.43), using (5.41) and (5.10) to get

$$\int_{x-\delta}^{x+\delta} p(x-y,t)|f(y)-f(x)|\, dy < \frac{\varepsilon}{3}. \tag{5.44}$$

Next, use (5.34) and the triangle inequality to obtain the following estimate for the last integral on the right–hand side of (5.43),

$$\int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\, dy \leqslant 2M \int_{x+\delta}^{\infty} p(x-y,t)\, dy. \tag{5.45}$$

Make the change of variables $\xi = y - x$ in the integral on the right–hand side of (5.45) to obtain

$$\int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x_o)|\, dy \leqslant 2M \int_{\delta}^{\infty} p(\xi,t)\, d\xi, \tag{5.46}$$

where we have also used the symmetry of the heat kernel (see (i) in Proposition 5.1.4). It follows from (5.46) and (5.19) in Lemma 5.1.6 that

$$\lim_{t\to 0^+} \int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\, dy = 0;$$

thus, there exists $\delta_1 > 0$ such that

$$0 < t < \delta_1 \Rightarrow \int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\, dy < \frac{\varepsilon}{3}. \tag{5.47}$$

Similar calculations to those leading to (5.47), using (5.20) in Lemma 5.1.6, can be used to show that there exists $\delta_2 > 0$ such that

$$0 < t < \delta_2 \Rightarrow \int_{-\infty}^{x-\delta} p(x-y,t)|f(y)-f(x)|\, dy < \frac{\varepsilon}{3}. \tag{5.48}$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. It then follows from (5.43), in conjunction with (5.44), (5.47) and (5.48), that

$$0 < t < \delta_3 \Rightarrow \int_{-\infty}^{\infty} p(x-y,t)|f(y) - f(x_o)|\ dy < \varepsilon.$$

We have therefore proved that

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} p(x-y,t)|f(y) - f(x)|\ dy = 0. \tag{5.49}$$

It follows from (5.49) and the estimate in (5.42) that

$$\lim_{t\to 0^+} |u(x,t) - f(x)| = 0,$$

which yields (5.14) and assertion (i) of Proposition 5.1.3 has been proved.

(ii) Assume that f has a jump discontinuity at x and put

$$f(x^+) = \lim_{y\to x^+} f(y) \quad \text{and} \quad f(x^-) = \lim_{y\to x^-} f(y). \tag{5.50}$$

Let $\varepsilon > 0$ be given. It follows from (5.50) that there exists $\delta > 0$ such that

$$x < y < x + \delta \Rightarrow |f(y) - f(x^+)| < \frac{\varepsilon}{3}, \tag{5.51}$$

and

$$x - \delta < y < x \Rightarrow |f(y) - f(x^-)| < \frac{\varepsilon}{3}. \tag{5.52}$$

Use the definition of $u(x,t)$ in (5.35) to write

$$u(x,t) - \frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{\infty} p(x-y,t)f(y)\ dy - \frac{1}{2}f(x^+) - \frac{1}{2}f(x^-),$$

and note that

$$\frac{1}{2} = \int_{-\infty}^{x_o} p(x_o - y,t)\ dy = \int_{x_o}^{\infty} p(x_o - y,t)\ dy, \tag{5.53}$$

by virtue of (5.10), (5.9) and the symmetry of the heat kernel (see (i) in Proposition 5.1.4). We therefore have that

$$\begin{aligned} u(x,t) \quad - \quad & \frac{f(x^+) + f(x^-)}{2} \\ &= \int_{-\infty}^{x} p(x-y,t)(f(y) - f(x^-))\ dy \\ &\quad + \int_{x_o}^{\infty} p(x-y,t)(f(y) - f(x^+))\ dy, \end{aligned}$$

so that

$$\begin{aligned}\left| u(x,t) - \frac{f(x^+)+f(x^-)}{2} \right| &\\ \leqslant \int_{-\infty}^{x} p(x-y,t)|f(y)-f(x^-)|\ dy & \qquad (5.54)\\ + \int_{x}^{\infty} p(x-y,t)|f(y)-f(x^+)|\ dy,&\end{aligned}$$

We re-write the last integral on the right–hand side of (5.54) as a sum of two integrals,

$$\begin{aligned}\int_{x}^{\infty} p(x-y, t)|f(y)-f(x^+)|\ dy &\\ = \int_{x}^{x+\delta} p(x-y,t)|f(y)-f(x^+)|\ dy & \qquad (5.55)\\ + \int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x^+)|\ dy,&\end{aligned}$$

where

$$\int_{x}^{x+\delta} p(x-y,t)|f(y)-f(x^+)|\ dy < \frac{\varepsilon}{3}\int_{x}^{x+\delta} p(x-y,t)\ dy < \frac{\varepsilon}{6}, \qquad (5.56)$$

by virtue of (5.52) and (5.53).

Similar calculations to those leading to (5.47) can be used to show that there exists $\delta_1 > 0$ such that

$$0 < t < \delta_1 \Rightarrow \int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x^+)|\ dy < \frac{\varepsilon}{3}. \qquad (5.57)$$

Combining (5.56) and (5.57), we obtain from (5.55) that

$$0 < t < \delta_1 \Rightarrow \int_{x}^{\infty} p(x-y,t)|f(y)-f(x^+)|\ dy < \frac{\varepsilon}{2}. \qquad (5.58)$$

Similarly, we can show that there exists $\delta_2 > 0$ such that

$$0 < t < \delta_2 \Rightarrow \int_{-\infty}^{x} p(x-y,t)|f(y)-f(x^-)|\ dy < \frac{\varepsilon}{2}. \qquad (5.59)$$

Thus, letting $\delta_3 = \min\{\delta_1, \delta_2\}$ we see that the conjunction of (5.58) and (5.59), together with (5.54), implies that

$$0 < t < \delta_3 \Rightarrow \left| u(x,t) - \frac{f(x^+)+f(x^-)}{2} \right| < \varepsilon.$$

We have therefore established (5.15) and the proof of part (ii) of Proposition 5.1.3 is now complete. ■

Example 5.1.8. Solve the initial value problem for the diffusion equation in (5.11), where

$$f(x) = \begin{cases} 1, & \text{if } -1 < x \leqslant 1; \\ 0, & \text{elsewhere.} \end{cases} \tag{5.60}$$

Solution: A sketch of the graph of the initial condition, f, is shown in Figure 5.1.3. Note that f has jump discontinuities at -1 and at 1.

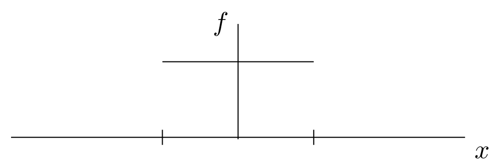

Figure 5.1.3: Initial Condition for Example 5.1.8

Using the formula in (5.35) we get that a solution to the initial value problem (5.11) with initial condition given in (5.60) is given by

$$u(x,t) = \int_{-1}^{1} p(x-y,t)\, dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

or

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-1}^{1} e^{-(x-y)^2/4Dt}\, dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.61}$$

Make the change variables $r = \dfrac{x-y}{\sqrt{4Dt}}$ in (5.61) to obtain

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\frac{x+1}{\sqrt{4Dt}}}^{\frac{x-1}{\sqrt{4Dt}}} e^{-r^2}\, dr, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

or

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+1}{\sqrt{4Dt}}} e^{-r^2}\, dr - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-1}{\sqrt{4Dt}}} e^{-r^2}\, dr, \tag{5.62}$$

for $x \in \mathbb{R}$ and $t > 0$.

Making use of the error function defined in (5.18), we can rewrite (5.62) as

$$u(x,t) = \frac{1}{2}\left[\operatorname{Erf}\left(\frac{x+1}{\sqrt{4Dt}}\right) - \operatorname{Erf}\left(\frac{x-1}{\sqrt{4Dt}}\right)\right], \tag{5.63}$$

for $x \in \mathbb{R}$ and $t > 0$. Figure 5.1.4 shows plots of the graph of $y = u(x,t)$, where $u(x,t)$ is as given in (5.63), for various values of t in the case $4D = 1$. A few interesting properties of the function u given in (5.63) are apparent by examining the pictures in Figure 5.1.4. First, the graph of $y = u(x,t)$ is smooth

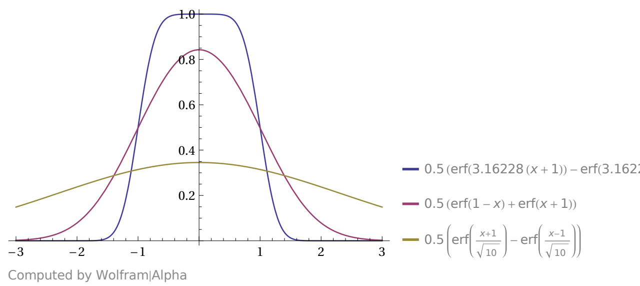

Figure 5.1.4: Sketch of Graph of $y = u(x,t)$ for $t = 0.1, 1, 10$

for all $t > 0$. Even though the initial temperature distribution, f, in (5.60) is not even continuous, the solution to the initial value problem (5.11) given in (5.63) is in fact infinitely differentiable as soon as the process gets going for $t > 0$. Secondly, the values, $u(x,t)$, of the function u given in (5.63)) are positive at all values of $x \in \mathbb{R}$ and $t > 0$. In particular, for values of x with $|x| > 1$, where the initial temperature is zero, the temperature rises instantly for $t > 0$. Thus, the diffusion model for heat propagation predicts that heat propagates with infinite speed. Thirdly, we see from the pictures in Figure 5.1.4 that

$$\lim_{t\to\infty} u(x,t) = 0, \qquad \text{for all } x \in \mathbb{R}. \tag{5.64}$$

□

5.1.2 Uniqueness for the Diffusion Equation

The observation (5.64) in Example 5.1.8 is true in general for solutions to the initial value problem in (5.11) for the case in which the initial condition, f, is square–integrable; that is,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \tag{5.65}$$

Observe that, for the function f in Example 5.1.8 satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 2,$$

so that the integrability condition in (5.65) holds true for the function in (5.60).

Before we establish that (5.64) is true for any solution of the initial value problem (5.11) in which the initial condition satisfies (5.65), we will first need to derive other properties of the function u given in (5.12).

Proposition 5.1.9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfying (5.65); that is,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty.$$

Put

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t) f(y) \, dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.66}$$

Then,

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx < \infty, \qquad \text{for all } t > 0, \tag{5.67}$$

and

$$\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x}(x,t) \right|^2 \, dx < \infty, \qquad \text{for all } t > 0. \tag{5.68}$$

Proof: Let u be given by (5.66), where f satisfies the condition in (5.65). Apply the Cauchy–Schwarz inequality (or Jensen's Inequality) to get

$$|u(x,t)|^2 \leqslant \int_{-\infty}^{\infty} p(x-y,t) |f(y)|^2 \, dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.69}$$

where we have also used (5.10) and the positivity of the heat kernel (see (ii) and (iii) in Proposition 5.1.4).

Integrate with respect to x on both sides of (5.69) to get

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx \leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x-y,t) |f(y)|^2 \, dy dx, \tag{5.70}$$

for $t > 0$. Interchanging the order of integration in the integral on the right–hand side of (5.70) we obtain

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx \leqslant \int_{-\infty}^{\infty} |f(y)|^2 \left\{ \int_{-\infty}^{\infty} p(x-y,t) \, dx \right\} dy, \tag{5.71}$$

for $t > 0$. It follows from (5.71) and (5.10) that

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx \leqslant \int_{-\infty}^{\infty} |f(y)|^2 \, dy, \qquad \text{for } t > 0, \tag{5.72}$$

Combining (5.72) and (5.65) then yields

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx < \infty, \qquad \text{for all } t > 0, \tag{5.73}$$

which is the condition in (5.67).

Next, differentiate u in (5.66) with respect to x to get

$$\frac{\partial u}{\partial x}(x,t) = -\int_{-\infty}^{\infty} \frac{(x-y)}{2Dt} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} f(y) \, dy,$$

so that

$$\frac{\partial u}{\partial x}(x,t) = -\int_{-\infty}^{\infty} p(x-y,t)\frac{(x-y)}{2Dt}f(y)\; dy, \tag{5.74}$$

for $x \in \mathbb{R}$ and $t > 0$.

Proceeding as in the first part of this proof, use the Cauchy–Schwarz inequality (or Jensen's inequality) to obtain from (5.74) that

$$\left|\frac{\partial u}{\partial x}(x,t)\right|^2 \leqslant \int_{-\infty}^{\infty} p(x-y,t)\frac{(x-y)^2}{4D^2t^2}|f(y)|^2\; dy, \tag{5.75}$$

for $x \in \mathbb{R}$ and $t > 0$.

Next, integrate on both sides of (5.75) with respect to x and interchange the order of integration to obtain

$$\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}(x,t)\right|^2\; dx \leqslant \frac{1}{4D^2t^2}\int_{-\infty}^{\infty}|f(y)|^2\int_{-\infty}^{\infty}(x-y)^2 p(x-y,t)\; dx\; dy, \tag{5.76}$$

for $t > 0$.

Observe that the inner integral in the right–hand side of (5.76) is simply the variance, $2Dt$, of the probability density function $p(x,t)$, so that

$$\int_{-\infty}^{\infty}(x-y)^2 p(x-y,t)\; dx = 2Dt, \quad \text{for all } y \in \mathbb{R} \text{ and } t > 0. \tag{5.77}$$

Putting together (5.76) and (5.77)

$$\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}(x,t)\right|^2\; dx \leqslant \frac{1}{2Dt}\int_{-\infty}^{\infty}|f(y)|^2\; dy, \quad \text{for } t > 0,$$

which implies (5.68) by virtue of (5.65). ■

We will next show that, if in addition to the integrability condition in (5.65) for the initial distribution, f, we also impose the conditions (5.67) and (5.68) on the initial value problem (5.11), then any solution must be of the form given in (5.12). This amounts to showing that the initial value problem (5.11) in which the initial condition satisfies (5.65), together with the integrability condition in (5.67) and (5.68), has a unique solution. We will need the estimate in the following lemma when we prove uniqueness.

Lemma 5.1.10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (5.65). Let

v be any solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & \text{for } x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}; \\ \displaystyle\int_{-\infty}^{\infty} |u(x,t)|^2\ dx < \infty, & \text{for all } t > 0; \\ \displaystyle\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx < \infty, & \text{for all } t > 0. \end{cases} \tag{5.78}$$

Then,

$$\int_{-\infty}^{\infty} |v(x,t)|^2\ dx \leqslant \int_{-\infty}^{\infty} |f(x)|^2\ dx, \quad \text{for } t \geqslant 0. \tag{5.79}$$

Proof: Let v denote any solution to the problem (5.78), where f satisfies the integrability condition in (5.65).

In order to establish (5.79), set

$$E(t) = \int_{-\infty}^{\infty} |v(x,t)|^2\ dx, \quad \text{for all } t \geqslant 0. \tag{5.80}$$

It follows from the integrability condition in (5.78) that $E(t)$ in (5.80) is well defined for all $t \geqslant 0$ as a real valued function, $E\colon [0,\infty) \to \mathbb{R}$. Note also that

$$E(0) = \int_{-\infty}^{\infty} |f(x)|^2\ dx, \tag{5.81}$$

by virtue of the initial condition in problem (5.78).

Next, observe that, since v satisfies the diffusion equation in (5.78), that is

$$v_t = Dv_{xx},$$

then E is differentiable and

$$E'(t) = \int_{-\infty}^{\infty} 2v(x,t)v_t(x,t)\ dx = 2D\int_{-\infty}^{\infty} v(x,t)v_{xx}(x,t)\ dx, \tag{5.82}$$

for $t > 0$.

We note that the integrability conditions in (5.78) imply that

$$\lim_{x\to\infty} v(x,t) = 0 \text{ and } \lim_{x\to-\infty} v(x,t) = 0, \quad \text{for } t > 0, \tag{5.83}$$

and

$$\lim_{x\to\infty} v_x(x,t) = 0 \text{ and } \lim_{x\to-\infty} v_x(x,t) = 0. \quad \text{for } t > 0, \tag{5.84}$$

Integrate by parts the last integral in (5.82) to get

$$E'(t) = \lim_{R\to\infty}\left[v(R,t)v_x(R,t) - v(-R,t)v_x(-R,t) - \int_{-R}^{R}(v_x(x,t))^2\ dx\right],$$

so that

$$E'(t) = -\int_{-\infty}^{\infty}\left|\frac{\partial v}{\partial x}(x,t)\right|^2\ dx, \quad \text{for } t>0, \tag{5.85}$$

by virtue of (5.83), (5.84) and the last integrability condition in (5.78).

Now, it follows from (5.85) that

$$E'(t) \leqslant 0, \quad \text{for all } t>0,$$

so that E is nondecreasing in t and therefore

$$E(t) \leqslant E(0), \quad \text{for all } t>0. \tag{5.86}$$

The estimate in (5.79) follows from (5.86) in view of (5.80) and (5.81). ■

Proposition 5.1.11. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (5.65). The problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & \text{for } x\in\mathbb{R},\ t>0; \\ u(x,0) = f(x), & \text{for } \ x\in\mathbb{R}; \\ \displaystyle\int_{-\infty}^{\infty}|u(x,t)|^2\ dx<\infty, & \text{for all } t>0; \\ \displaystyle\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx<\infty, & \text{for all } t>0, \end{cases} \tag{5.87}$$

has at most one solution.

Proof: Let v be any solution of the problem in (5.87) and let u be given by (5.66). It follows from Proposition 5.1.3 and Proposition 5.1.9 that u solves problem (5.87). Put

$$w(x,t) = v(x,t) - u(x,t), \quad \text{for } x\in\mathbb{R} \text{ and } t>0. \tag{5.88}$$

It follows from the linearity of the differential equation in (5.87) that w also solves the diffusion equation; indeed,

$$w_t = v_t - u_t = Dv_{xx} - Du_{xx} = D(v_{xx}-u_{xx}) = Dw_{xx}.$$

The function w defined in (5.88) also satisfies the integrability condition in problem (5.87) ; in fact, by the triangle inequality,

$$|w(x,t)| \leqslant |v(x,t)| + |u(x,t)|,$$

so that

$$|w(x,t)|^2 \leqslant |v(x,t)|^2 + 2|v(x,t)| \cdot |u(x,t)| + |u(x,t)|^2, \tag{5.89}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Next, use the inequality

$$2ab \leqslant a^2 + b^2, \qquad \text{for } a, b \in \mathbb{R},$$

in (5.89) to get

$$|w(x,t)|^2 \leqslant 2\left[|v(x,t)|^2 + |u(x,t)|^2\right], \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.90}$$

Integrating on both sides of (5.90) with respect to x we then obtain that

$$\int_{-\infty}^{\infty} |w(x,t)|^2\, dx \leqslant 2\left[\int_{-\infty}^{\infty} |v(x,t)|^2\, dx + \int_{-\infty}^{\infty} |u(x,t)|^2\, dx\right], \quad \text{for } t > 0,$$

so that

$$\int_{-\infty}^{\infty} |w(x,t)|^2\, dx < \infty, \quad \text{for } t > 0,$$

since both u and v satisfy the integrability conditions in problem (5.87). Similarly, we can show that

$$\int_{-\infty}^{\infty} |w_x(x,t)|^2\, dx < \infty, \quad \text{for } t > 0.$$

Now, observe that, since both v and u satisfy the initial condition in problem (5.87,

$$w(x,0) = v(x,0) - u(x,0) = f(x) - f(x) = 0, \quad \text{for all } x \in \mathbb{R},$$

so that w is a solution of problem (5.78) in which the initial condition is the constant function 0, it follows from the estimate (5.79) in Lemma 5.1.10 that

$$\int_{-\infty}^{\infty} |w(x,t)|^2\, dx \leqslant 0, \quad \text{for } t \geqslant 0,$$

from which we get that

$$\int_{-\infty}^{\infty} |w(x,t)|^2\, dx = 0, \quad \text{for } t \geqslant 0. \tag{5.91}$$

It follows from (5.91) and the continuity of w that

$$w(x,t) = 0, \quad \text{for all } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

so that

$$v(x,t) = u(x,t), \quad \text{for all } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

in view of the definition of w in (5.88). Hence, any solution to the problem in (5.87) must be that given by (5.66). ■

We will next show that, if u is any solution of problem (5.87), where f satisfies the integrability condition

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty, \tag{5.92}$$

then

$$\lim_{t\to\infty} u(x,t) = 0, \quad \text{for all } x \in \mathbb{R}. \tag{5.93}$$

To see why this is the case, apply Proposition 5.1.11 to write

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} f(y) \, dy,$$

for all $x \in \mathbb{R}$ and $t > 0$, from which we get that

$$|u(x,t)| \leqslant \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} |f(y)| \, dy, \tag{5.94}$$

for all $x \in \mathbb{R}$ and $t > 0$. Next, square on both sides of (5.94) and apply the Cauchy–Schwarz inequality to get

$$|u(x,t)|^2 \leqslant \frac{1}{\sqrt{8\pi Dt}} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2Dt}}{\sqrt{2\pi Dt}} \, dy \int_{-\infty}^{\infty} |f(y)|^2 \, dy, \tag{5.95}$$

where

$$\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2Dt}}{\sqrt{2\pi Dt}} \, dy = 1. \tag{5.96}$$

Combining (5.95) and (5.96), we then get

$$|u(x,t)|^2 \leqslant \frac{1}{\sqrt{8\pi Dt}} \int_{-\infty}^{\infty} |f(y)|^2 \, dy, \tag{5.97}$$

for $x \in \mathbb{R}$ and $t > 0$.

It follows from (5.92) and (5.97) that

$$\lim_{t\to\infty} |u(x,t)|^2 = 0, \quad \text{for all } x \in \mathbb{R},$$

which implies (5.93).

5.2 Solving the Dirichlet Problem in the Unit Disk

The goal of this section is to construct a solutions to the boundary value problem for the two–dimensional Laplacian

$$\begin{cases} u_{xx} + u_{yy} &= 0 \qquad \text{in } D_1; \\ u(x,y) &= h(x,y), \quad \text{for } (x,y) \in \partial D_1, \end{cases} \tag{5.98}$$

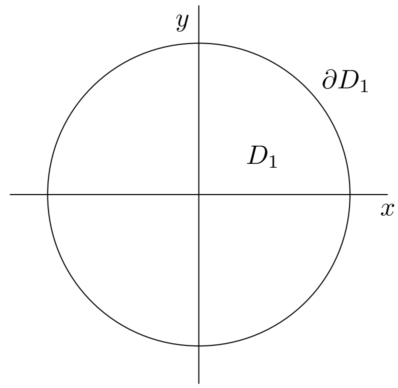

Figure 5.2.5: Unit Disk in $\mathbb{R}^2$

where $D_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is the unit disk in $\mathbb{R}^2$, and h is a given function that is continuous in a neighborhood of the unit circle ∂D_1. Thus, we would like to find a function, u, that is harmonic in D_1 and that takes on the values given by a continuous function, h, on the boundary of D_1.

The procedure that we will follow here construct a solution to the Dirichlet problem in (5.2) illustrates an approach that has been successful in the solution of boundary value problems for linear PDEs over domains with simple geometry.

- First, in view of the radial symmetry of the domain, we will express problem (5.2) in polar coordinate r and θ.

- Next, we look for a special type of solutions that are products of a function of r and a function of θ. In other words, we look for solutions in which the variables **separate**; this is known as the method of **separation of variables**.

- When looking solutions that are nonzero over the domain by means of separation of variables, we are invariable lead to an **eigenvalue problem**. Solution of the eigenvalue problem leads to a family of solutions in one (or both of the variables), called **eigenfunctions**. These eigenfunctions generate a special family of solutions.

- We will then use the principle of superposition to construct linear combinations of the eigenfunction solutions. We hope that a sequence of these linear combinations will converge to a function that solves the PDE in (5.98) and satisfies the boundary condition in that problem; this method is usually referred to as **eigenfunctions expansion**.

5.2.1 Separation of Variables

In view of the radial symmetry of the domain (see Figure 5.2.5), we will treat the problem in polar coordinates, (r, θ), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

We will also exploit the linearity of the PDE and the boundary condition in (5.98) and use the principle of superposition to construct a solution of the problem by superposing simple solutions of the problem. The strategy then is to, first, find a special class of functions of r and θ that solve Laplace's equation, and then use sums of those solutions to construct a solution that also satisfies the boundary condition.

We begin by expressing the BVP (5.98) in polar coordinates:

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & r > 0, -\pi < \theta \leqslant \pi; \\ v(1,\theta) = h(\cos\theta, \sin\theta), & -\pi < \theta \leqslant \pi, \end{cases} \tag{5.99}$$

where we have set

$$v(r,\theta) = u(r\cos\theta, r\sin\theta).$$

(See Problem 3 in Assignment #11). We will denote $h(\cos\theta, \sin\theta)$ by $g(\theta)$, where $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous periodic function of period 2π. We can then rewrite the BVP in (5.99) as

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & r > 0, -\pi < \theta \leqslant \pi; \\ v(1,\theta) = g(\theta), & -\pi < \theta \leqslant \pi. \end{cases} \tag{5.100}$$

We start out by looking for special solutions of the PDE in (5.100) of the form

$$v(r,\theta) = f(r)z(\theta), \quad \text{for } r \geqslant 0 \text{ and } -\pi < \theta < \pi, \tag{5.101}$$

where $f\colon [0,\infty) \to \mathbb{R}$ is a continuous functions that is C^2 in $(0,\infty)$, and $z\colon \mathbb{R} \to \mathbb{R}$ is a C^2, periodic function of period 2π. We can therefore compute the partial derivatives,

$$\frac{\partial v}{\partial r}(r,\theta) = f'(r)z(\theta), \quad r > 0, -\pi < \theta \leqslant \pi;$$

$$\frac{\partial^2 v}{\partial r^2}(r,\theta) = f''(r)z(\theta), \quad r > 0, -\pi < \theta \leqslant \pi;$$

$$\frac{\partial^2 v}{\partial \theta^2}(r,\theta) = f(r)z''(\theta), \quad r > 0, -\pi < \theta \leqslant \pi,$$

and substitute them into the PDE in (5.100) to obtain

$$f''(r)z(\theta) + \frac{1}{r}f'(r)z(\theta) + \frac{1}{r^2}f(r)z''(\theta) = 0, \quad \text{for } r > 0, -\pi < \theta \leqslant \pi. \tag{5.102}$$

Assuming that $v(r,\theta)$ is not zero for all values of r and θ, and dividing on both sides of (5.102) by $v(r,\theta)$ as given in (5.101), we obtain

$$\frac{f''(r)}{f(r)}+\frac{1}{r}\frac{f'(r)}{f(r)}+\frac{1}{r^2}\frac{z''(\theta)}{z(\theta)}=0,\qquad \text{for } r>0, -\pi<\theta\leqslant\pi. \tag{5.103}$$

Multiplying on both sides of the equation in (5.103) by r^2, we notice that that equation can be written in such a way that the functions that depend only on r are on one side of the equation and those that depend only on θ are on the other side of the equation:

$$r^2\frac{f''(r)}{f(r)}+r\frac{f'(r)}{f(r)}=-\frac{z''(\theta)}{z(\theta)},\qquad \text{for } r>0, -\pi<\theta\leqslant\pi. \tag{5.104}$$

Since (5.104) holds true for all values of r and θ in $(0,1)$ and $(-\pi,\pi]$, respectively, it follows from (5.104) that each side of the equation in (5.104) must be equal to a constant.[2] Call that constant λ so that

$$r^2\frac{f''(r)}{f(r)}+r\frac{f'(r)}{f(r)}=-\frac{z''(\theta)}{z(\theta)}=\lambda,\qquad \text{for } r>0, -\pi<\theta\leqslant\pi. \tag{5.105}$$

The expression in (5.105) leads to two ordinary differential equations

$$-z''(\theta)=\lambda z(\theta),\qquad \text{for } -\pi<\theta\leqslant\pi, \tag{5.106}$$

and

$$r^2f''(r)+rf'(r)=\lambda f(r),\qquad \text{for } r>0. \tag{5.107}$$

The requirement that the function g in (5.100) be periodic of period 2π yields the following conditions for z:

$$z(-\pi)=z(\pi)\quad\text{and}\quad z'(-\pi)=z'(\pi); \tag{5.108}$$

in other words, we will assume that z can be extended to a C^1 periodic function defined on $\mathbb{R}$ with period two 2π. Putting together (5.106) and (5.108) yields the following **two–point boundary value problem**:

$$\left\{\begin{array}{rcll} -z''(\theta) & = & \lambda z(\theta), & \text{for } -\pi<\theta<\pi;\\ z(-\pi) & = & z(\pi); & \\ z'(-\pi) & = & z'(\pi). & \end{array}\right. \tag{5.109}$$

[2] To see why this assertion is true, pick θ_o in $(-\pi,\pi]$ such that $z(\theta_o)\neq 0$; then, by virtue of (5.104), $r^2\frac{f''(r)}{f(r)}+r\frac{f'(r)}{f(r)}=-\frac{z''(\theta_o)}{z(\theta_o)}$, for all $r>0$; so that the left–hand side of (5.104) is constant. Similarly, for fixed r_o in $(0,1)$ with $f(r_o)\neq 0$, (5.104) implies that $\frac{z''(\theta)}{z(\theta)}=-r_o^2\frac{f''(r_o)}{f(r_o)}-r_o\frac{f'(r_o)}{f(r_o)}$, for all θ in $(-\pi,\pi]$, so that the right–hand side of (5.104) must also be constant.

5.2.2 An Eigenvalue Problem

Observe that the constant function $z(\theta) = 0$, for all values of θ, solves two–point BVP in (5.109); we shall refer to this solution as the **trivial** solution. We are interested in **nontrivial** solutions of (5.109); otherwise, the special solutions in (5.101) of the BVP in (5.100) that we are seeking would all be the zero function. These solutions will not be helpful in the construction of a solution of the BVP in (5.100) for arbitrary (nonzero) boundary conditions. We will see shortly that the answer to the question of whether or not the two–point BVP in (5.109) has nontrivial solutions depends on the value of λ in the ODE in that problem. In fact, there is a certain set of values of λ for which (5.109) has nontrivial solutions; for the rest of the values of λ the two–point BVP (5.109) has only the trivial solution.

Definition 5.2.1 (Eigenvalues and Eigenfunctions)**.** A value of λ in (5.109) for which the two–point BVP in (5.109) has a nontrivial is called an **eigenvalue** of the BVP; a corresponding nontrivial solution is called an **eigenfunction**.

We will next compute the eigenvalues and eigenfunctions of the two–point BVP in (5.109). Before we proceed with the calculations, it will be helpful to know that the eigenvalues of (5.109) must be nonnegative. We state that fact in the following proposition.

Proposition 5.2.2. Assume that the two–point BVP (5.109) has nontrivial solution. Then, $\lambda \geqslant 0$.

Proof: Let z be a nontrivial solution of (5.109). Multiply the ODE in (5.109) by z and integrate from $-\pi$ to π to get

$$-\int_{-\pi}^{\pi} z''(\theta)z(\theta)\ d\theta = \lambda \int_{-\pi}^{\pi} z(\theta)z(\theta)\ d\theta. \tag{5.110}$$

Use integration by parts to evaluate the left–most integral in (5.110) to get

$$\int_{-\pi}^{\pi} z''(\theta)z(\theta)\ d\theta = z(\theta)z'(\theta)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} z'(\theta)z'(\theta)\ d\theta,$$

so that, in view of the boundary conditions in (5.109),

$$\int_{-\pi}^{\pi} z''(\theta)z(\theta)\ d\theta = -\int_{-\pi}^{\pi} [z'(\theta)]^2\ d\theta. \tag{5.111}$$

Substituting the result in (5.111) into the left–hand side of (5.110) then yields

$$\int_{-\pi}^{\pi} [z'(\theta)]^2\ d\theta = \lambda \int_{-\pi}^{\pi} [z(\theta)]^2\ d\theta. \tag{5.112}$$

Since z is a nontrivial solution of the two–point BVP in (5.109), it follows that

$\int_{-\pi}^{\pi} [z(\theta)]^2\, d\theta > 0$. We can therefore solve (5.112) for λ to obtain

$$\lambda = \frac{\int_{-\pi}^{\pi} [z'(\theta)]^2\, d\theta}{\int_{-\pi}^{\pi} [z(\theta)]^2\, d\theta},$$

which shows that λ is nonnegative. ■

In view of the result of Proposition 5.2.2, it suffices to look for nontrivial solutions of (5.109) for either $\lambda = 0$ or $\lambda > 0$.

For the case in which $\lambda = 0$ in (5.109), the ODE in (5.109) becomes

$$z''(\theta) = 0,$$

which has general solution

$$z(\theta) = c_1\theta + c_2, \tag{5.113}$$

for arbitrary constants c_1 and c_2.

Applying the first boundary condition to z given in (5.113) yields

$$-\pi c_1 + c_2 = \pi c_2 + c_2,$$

from which we get that $2\pi c_1 = 0$, so that $c_1 = 0$. It then follows from (5.113) any solution of the BVP in (5.109) with $\lambda = 0$ must be constant:

$$z(\theta) = c, \quad \text{for all } \theta. \tag{5.114}$$

In particular, if $c \neq 0$ in (5.114), $z(\theta) = c$ for all θ is a nontrivial solution of the two–point BVP (5.109). Consequently, $\lambda = 0$ is an eigenvalue of (5.109). For future reference, we shall denote this eigenvalue by λ_o, so that

$$\lambda_o = 0, \tag{5.115}$$

and we shall pick the special eigenfunction

$$\varphi_o(\theta) = 1, \quad \text{for all } \theta, \tag{5.116}$$

and note that any solution of the BVP in (5.109) for λ_o is a constant multiple of φ_o given in (5.116); so that

$$z_o(\theta) = a_o, \quad \text{for all } \theta, \tag{5.117}$$

where a_o denotes a real constant, represents all solutions of the two–point BVP in (5.116) corresponding to to the eigenvalue $\lambda_o = 0$.

Next, we look for positive eigenvalues of the BVP in (5.109). For the case in which $\lambda > 0$ in (5.109), the general solution of the ODE in (5.109) is

$$z(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta), \quad \text{for all } \theta, \tag{5.118}$$

and arbitrary constants c_1 and c_2, so that

$$z'(\theta) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\theta), \quad \text{for all } \theta. \tag{5.119}$$

Imposing the the boundary conditions in (5.109) to the functions given in (5.118) and (5.119) yields the system of equations

$$\left\{ \begin{array}{rcl} c_1\cos(-\sqrt{\lambda}\pi) + c_2\sin(-\sqrt{\lambda}\pi) & = & c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi); \\ -c_1\sqrt{\lambda}\sin(-\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(-\sqrt{\lambda}\pi) & = & -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi); \end{array} \right. \tag{5.120}$$

thus, dividing the second equation in (5.120) by $\sqrt{\lambda}$ since $\lambda > 0$, and using the fact that cos is even and sin is odd,

$$\left\{ \begin{array}{rcl} c_1\cos(\sqrt{\lambda}\pi) - c_2\sin(-\sqrt{\lambda}\pi) & = & c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi); \\ c_1\sin(\sqrt{\lambda}\pi) + c_2\cos(\sqrt{\lambda}\pi) & = & -c_1\sin(\sqrt{\lambda}\pi) + c_2\cos(\sqrt{\lambda}\pi), \end{array} \right.$$

from which we get that

$$\left\{ \begin{array}{rcl} 2c_2\sin(\sqrt{\lambda}\pi) & = & 0; \\ 2c_1\sin(\sqrt{\lambda}\pi) & = & 0. \end{array} \right. \tag{5.121}$$

Since we are looking for nontrivial solutions of (5.109), we require that c_1 and c_2 in (5.118) are not both zero. Consequently, we obtain from (5.121) that

$$\sin(\sqrt{\lambda}\pi) = 0. \tag{5.122}$$

Solutions to the trigonometric equation in (5.122) are given by

$$\sqrt{\lambda}\pi = n\pi \tag{5.123}$$

where n is an integer. It follows from (5.123) that the positive eigenvalues of the BVP in (5.109) are given by

$$\lambda = n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.124}$$

We will denote the positive eigenvalues of the BVP (5.109) in (5.124) by λ_n, for $n = 1, 2, 3, \ldots$, so that

$$\lambda_n = n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.125}$$

We will denote the corresponding eigenfunctions by z_n. These are linear combinations of $\cos(n\theta)$ and $\sin(n\theta)$, so that

$$z_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta) \quad \text{for } n = 1, 2, 3, \ldots, \text{ and } \theta \in \mathbb{R}, \tag{5.126}$$

where a_n and b_n, for $n = 1, 2, 3\ldots$, are real constants.

We shall put together the results in (5.115), (5.117), (5.125) and (5.126) in the following proposition:

Proposition 5.2.3 (Eigenvalues and eigenfunctions of BVP (5.109))**.** The eigenvalues of the two–point BVP (5.109) are given by

$$\lambda_n = n^2, \quad \text{for } n = 0, 1, 2, 3, \ldots, \tag{5.127}$$

with corresponding eigenfunctions of the form

$$z_o(\theta) = a_o, \quad \text{for all } \theta \in \mathbb{R},$$

and

$$z_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta) \quad \text{for } n = 1, 2, 3, \ldots, \text{ and } \theta \in \mathbb{R},$$

where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are real constants.

We the values for λ given in (5.127) we now proceed to solve the ODE in (5.107) for the radial component of the special solutions to the BVP in (5.100) of the form given in (5.101); namely,

$$r^2 f''(r) + r f'(r) = n^2 f(r), \quad \text{for } r > 0 \text{ and } n = 0, 1, 2, \ldots \tag{5.128}$$

We shall first solve (5.128) for the case $n = 0$. In this case the equation becomes

$$r f''(r) + f'(r) = 0, \quad \text{for } r > 0, \tag{5.129}$$

where we have divided by $r > 0$. Observe that the equation in (5.129) can be written as

$$\frac{d}{dr}[r f'(r)] = 0, \quad \text{for } r > 0,$$

which can be integrated to yield

$$r f'(r) = c_1, \quad \text{for } r > 0,$$

and some constant c_1, or

$$f'(r) = \frac{c_1}{r}, \quad \text{for } r > 0, \tag{5.130}$$

and some constant c_1. Integrating the equation in (5.130) then yields

$$f(r) = c_1 \ln(r) + c_2, \quad \text{for } r > 0, \tag{5.131}$$

and some constants c_1 and c_2. Observe that, if $c_1 \neq 0$ in (5.131), the the function f given a (5.131) is not unbounded as $r \to 0^+$. Thus, since we are looking for C^2 functions defined in the closure of the unit disk, $\overline{D_1}$, we must set c_1 equal to 0. This is equivalent to imposing the following boundary condition on f:

$$\lim_{r \to 0^+} f(r) \text{ exists.} \tag{5.132}$$

Hence, it follows from (5.131) and (5.132) that, for $n = 0$, a solution for (5.128) is given by

$$f(r) = c, \quad \text{for all } r, \tag{5.133}$$

is a solution, for some constant c. Taking $c = 1$ in (5.133) we get the solution to (5.128) corresponding to $n = 0$:

$$f_o(r) = 1, \quad \text{for all } r. \tag{5.134}$$

Next, consider the case $n \geqslant 1$ in (5.128). In this case the differential equation in (5.128) is an ODE of Euler type:

$$r^2 f''(r) + r f'(r) - n^2 f(r) = 0, \quad \text{for } r > 0. \tag{5.135}$$

The ODE in (5.135) can be solved by looking for solutions of the form

$$f(r) = r^q, \quad \text{for } r > 0 \tag{5.136}$$

and some real number q.

Taking derivatives of f in (5.136) and substituting into (5.136) yields

$$r^2 q(q-1) r^{q-2} + r q r^{q-1} - n^2 r^q = 0, \quad \text{for } r > 0,$$

or

$$q(q-1) r^q + q r^q - n^2 r^q = 0, \quad \text{for } r > 0,$$

or

$$[q(q-1) + q - n^2] r^q = 0, \quad \text{for } r > 0. \tag{5.137}$$

It follows from (5.137) that

$$q(q-1) + q - n^2 = 0,$$

or

$$q^2 - n^2 = 0,$$

or

$$(q+n)(q-n) = 0,$$

from which we get that

$$q = \pm n. \tag{5.138}$$

It follows from (5.137) and (5.138) that

$$f_{-n}(r) = r^{-n} \quad \text{and} \quad f_n(r) = r^n, \quad \text{for } r > 0. \tag{5.139}$$

In view of the boundary condition in (5.132), we take the second solution in (5.139),

$$f_n(r) = r^n, \quad \text{for all } r \text{ and } n = 1, 2, 3, \ldots. \tag{5.140}$$

Putting together (5.140), (5.134), (5.126), (5.117), and (5.101), we conclude that we have found an infinite collection of solutions to the PDE in (5.100); namely,

$$v_o(r, \theta) = a_o, \quad \text{for all } r \text{ and } \theta; \tag{5.141}$$

$$v_n(r, \theta) = r^n [a_n \cos(n\theta) + b_n \sin(n\theta)], \quad \text{for all } r \text{ and } \theta, \tag{5.142}$$

where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are real constants.

5.2.3 Expansion in Terms of Eigenfunctions

None of the functions in (5.141) and (5.142) by itself will satisfy the general boundary condition in (5.100). We can, however, attempt to construct a solution to (5.100) adding all of them together; in other words, by applying the principle of superposition:

$$v(r, \theta) = a_o + \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)], \ 0 \leqslant r < 1, \ -\pi < \theta \leqslant \pi. \quad (5.143)$$

provided the series in (5.143) converges to a C^2 function.

Let's assume for the moment that the series in (5.143) converges also for $r = 1$, so that we can apply the boundary condition in (5.100) to get

$$a_o + \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)] = g(\theta), \qquad \text{for } -\pi < \theta \leqslant \pi. \qquad (5.144)$$

Assuming for the moment that the series on the left–hand side of (5.144) converges in such a way that it can be integrated term by term, we can compute the values of the coefficients a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, in terms of the function g by means of the following integration facts:

$$\int_{-\pi}^{\pi} \sin(n\theta) \cos(m\theta)\ d\theta = 0, \qquad \text{for all } m, n = 1, 2, 3, \ldots; \qquad (5.145)$$

$$\int_{-\pi}^{\pi} \cos(n\theta) \cos(m\theta)\ d\theta = \begin{cases} 0, & \text{if } m \neq n; \\ \pi, & \text{if } m = n; \end{cases} \qquad (5.146)$$

and

$$\int_{-\pi}^{\pi} \sin(n\theta) \sin(m\theta)\ d\theta = \begin{cases} 0, & \text{if } m \neq n; \\ \pi, & \text{if } m = n. \end{cases} \qquad (5.147)$$

Indeed, integrating on both sides of (5.144) from $-\pi$ to π we get, assuming that the series in (5.144) can be integrated term by term,

$$2\pi a_o = \int_{-\pi}^{\pi} g(\theta)\ d\theta,$$

from which we get

$$a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\ d\theta; \qquad (5.148)$$

thus, a_o is the average value of g over the interval $(-\pi, \pi]$.

Next, multiply the equation in (5.144) on both sides by $\cos(m\theta)$ to obtain

$$a_o \cos m\theta + \sum_{n=0}^{\infty} [a_n \cos n\theta \cos m\theta + b_n \sin n\theta \cos m\theta] = g(\theta) \cos m\theta. \qquad (5.149)$$

Then, integrate on both sides of (5.149) with respect to θ from $-\pi$ to π, and use the identities in (5.145) and (5.146) to get

$$\pi a_m = \int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\ d\theta,$$

from which we get

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\ d\theta, \qquad \text{for } m = 1, 2, 3, \ldots. \tag{5.150}$$

Similar calculations (this time multiplying the equation in (5.144) on both sides by $\sin(m\theta)$, integrating from $-\pi$ to π, and using the integral identities in (5.145) and (5.147)) lead to

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\ d\theta, \qquad \text{for } m = 1, 2, 3, \ldots. \tag{5.151}$$

The numbers defined in (5.148), (5.150) and (5.151) are called the **Fourier coefficients** of the 2π–periodic function g. Note that the Fourier coefficients of g are defined whenever g is **absolutely integrable** over the interval $[-\pi, \pi]$.

Definition 5.2.4 (Absolute Integrability)**.** A function $g\colon [-\pi, \pi] \to \mathbb{R}$ is said to be absolutely integrable over $[-\pi, \pi]$ whenever

$$\int_{-\pi}^{\pi} |g(\theta)|\ d\theta < \infty. \tag{5.152}$$

Note that g doesn't have to be continuous for (5.152) for (5.152). For instance, if g is bounded and piece–wise continuous then (5.152) holds; indeed, suppose that g piece–wise continuous and

$$|g(\theta)| \leqslant M, \qquad \text{for } \theta \in [-\pi, \pi],$$

and some positive constant M; then

$$\int_{-\pi}^{\pi} |g(\theta)|\ d\theta \leqslant \int_{-\pi}^{\pi} M\ d\theta = 2\pi M < \infty.$$

Notation 5.2.5. We will denote the integral in (5.152) by $\|g\|_{L^1}$; so that

$$\|g\|_{L^1} = \int_{-\pi}^{\pi} |g(\theta)|\ d\theta. \tag{5.153}$$

If the integral in (5.153) ia understood as the Lebesgue integral, and $\|g\|_{L^1} < \infty$ we will say that g is an L^1 function and write $g \in L^1(-\pi, \pi)$. We shall refer to $\|g\|_{L^1}$ as the L^1 **norm of** $g \in L^1(-\pi, \pi)$.

The existence of the Fourier coefficients of g in (5.148), (5.150) and (5.151) is guaranteed for absolutely integrable 2π–periodic functions, g, or for $g \in L^1(-\pi, \pi)$. This is the content of the following proposition.

Proposition 5.2.6 (Existence of the Fourier Coefficients)**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.148) and (5.150), and b_n, for $n = 1, 2, 3, \ldots$, be as in (5.151), where $g \in L^1(-\pi, \pi)$. Then,

$$|a_n| \leqslant \frac{1}{\pi}\|g\|_{L^1}, \qquad \text{for } n = 0, 1, 2, 3, \ldots; \tag{5.154}$$

and

$$|b_n| \leqslant \frac{1}{\pi}\|g\|_{L^1}, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{5.155}$$

Proof: The estimates in (5.154) and (5.155) follow from properties of the integral. For a_o, we get from (5.148) that

$$|a_o| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\theta)|\ d\theta,$$

so that, using the definition of the L^1 norm of g in (5.153),

$$|a_o| \leqslant \frac{1}{2\pi}\|g\|_{L^1} \leqslant \frac{1}{\pi}\|g\|_{L^1}.$$

For $n = 1, 2, 3, \ldots$ we obtain from (5.150) that

$$\begin{aligned} |a_n| &\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |g(\theta)|\ |\cos(n\theta)|\ d\theta \\ &\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |g(\theta)|\ d\theta, \end{aligned}$$

since $|\cos(n\theta)| \leqslant 1$ for all θ and all n, which yields (5.154). Similar calculations lead to (5.155). ■

It follows from Proposition 5.2.6 that the sequences of Fourier coefficients, (a_n) and (b_n), of g are bounded by a constant depending on the L^1 norm of g. In fact, it can be shown that the Fourier coefficients of an L^1 2π–periodic functions tend to 0 as n goes to infinity; this is known as the Riemann–Lebesgue Lemma.

Proposition 5.2.7 (Riemann–Lebesgue Lemma)**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.148) and (5.150), and b_n, for $n = 1, 2, 3, \ldots$, be as in (5.151), where $g \in L^1(-\pi, \pi)$. Then,

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

For a proof of the Riemann-Lebesgue Lemma, see [Tol62].

We will next use the result of Proposition 5.2.6 to prove that the series defining the function v in (5.143) converges in D_1.

Proposition 5.2.8 (Point–wise Convergence of Series in (5.143))**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.148) and (5.150), and b_n, for $n = 1, 2, 3, \ldots$ be as in (5.151), where $g \in L^1(-\pi, \pi)$. Then, the series defining v in (5.143) converges absolutely in D_1.

Proof: The conclusion will follow by comparing with the geometric series since $0 \leqslant r < 1$ and

$$|r^n a_n \cos(n\theta)| \leqslant r^n |a_n| \leqslant \frac{\|g\|_{L^1}}{\pi} r^n, \qquad \text{for all } n,$$

where we have used the estimate (5.154) in Proposition 5.2.6. Similarly, using (5.155) in Proposition 5.2.6,

$$|r^n b_n \sin(n\theta)| \leqslant \frac{\|g\|_{L^1}}{\pi} r^n,$$

for all n. ■

Proposition 5.143) allows us to conclude that the function v given in (5.143) is well defined. However, in order to prove that that function is harmonic in D_1, we have to be able to differentiate the series term by term. This would be possible, for instance, if we knew that the series on the right–hand–side of (5.143), and the series for the partial derivatives

$$\sum_{n=0}^{\infty} n r^n [-a_n \sin(n\theta) + b_n \cos(n\theta)],$$

$$-\sum_{n=0}^{\infty} n^2 r^n [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

$$\sum_{n=0}^{\infty} n r^{n-1} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

and

$$\sum_{n=0}^{\infty} n(n-1) r^{n-2} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

converge uniformly. However, we do not know that at this point. In order to answer these questions, though, we will have to make further assumptions on g. Before we deal with these questions, we will first answer the question of when the trigonometric series on the left–hand side of the equation in (5.144) converges uniformly. Uniform convergence will justify the term–by–term integration that was done in order to obtain the formulas in (5.148), (5.150) and (5.151). We will denote the trigonometric series on the left–hand side of the equation in (5.144) by $\widehat{g}(\theta)$, so that

$$\widehat{g}(\theta) = a_o + \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)], \qquad \text{for } -\pi \leqslant \theta \leqslant \pi, \tag{5.156}$$

where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of g. Let $(\widehat{g}_n(\theta))$ denote the sequence of partial sums of the series in (5.156) so that

$$\widehat{g}_n(\theta) = a_o + \sum_{k=0}^{k} [a_k \cos(k\theta) + b_k \sin(k\theta)], \quad \text{for } -\pi \leqslant \theta \leqslant \pi. \tag{5.157}$$

Definition 5.2.9 (Uniform Convergence)**.** We say that sequence of functions, $(\widehat{g}_n)$, defined in (5.157) converges uniformly to g in $[-\pi, \pi]$ if

$$\lim_{n\to\infty} \max_{-\pi\leqslant\theta\leqslant\pi} |\widehat{g}_n(\theta) - g(\theta)| = 0.$$

The following proposition gives a sufficient condition for the trigonometric series in (5.156) to converge uniformly to g.

Theorem 5.2.10 (Uniform Convergence)**.** Assume that $g \colon \mathbb{R} \to \mathbb{R}$ is a continuous 2π–periodic function; assume also that g is piece–wise differentiable with $g' \colon \mathbb{R} \to \mathbb{R}$ piecewise continuous. Let $(\widehat{g}_n)$ be the sequence of trigonometric functions defined in (5.156), where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of g. Then, $(\widehat{g}_n)$ converges uniformly to g in $[-\pi, \pi]$ as $n \to \infty$.

A proof of Theorem 5.2.10 may be found in [Tol62, pp. 80-81]. The idea of the proof is to derive the estimate

$$\sum_{k=0}^{\infty} (|a_k| + |b_k|) < \infty, \tag{5.158}$$

for the Fourier coefficients of a piece–wise C^1, 2π–periodic function g. The uniform convergence of the series in (5.156) then follows from the observation that sum of the absolute values of the terms in the series on the right–hand side of (5.156) is bounded above by the series on the left–hand side of (5.158). The uniform convergence of the series in (5.156) is then a consequence of Weierstrass Majorization Test, or Weierstrass M–Test for uniform convergence, (see, for example, [Rud53, Theorem 7.10, pg. 119]).

Definition 5.2.11 (Piece–wise C^1)**.** We say that $g \colon \mathbb{R} \to \mathbb{R}$ is piecewise C^1 if it is differentiable, and its derivative is piece–wise continuous.

Let's assume for the moment that g is a 2π–periodic, piece–wise C^1 function. It then follows from Theorem 5.2.10 that the Fourier series on the left–hand side of (5.144), where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of g, converges uniformly to the right–hand side of the equation. This justifies the term–by–term integration of the series that lead to the formulas for the Fourier coefficients in (5.148), (5.150) and (5.151) by virtue of the following theorem form Analysis:

Theorem 5.2.12 (Term–by–Term Integration)**.** Let (u_k) be a sequence of continuous functions over a closed and bounded interval, $[a, b]$. Assume that the series

$$\sum_{k=1}^{\infty} u_k$$

converges uniformly to f. Then, f is continuous on $[a, b]$, and

$$\int_a^b f(x)\ dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)\ dx,$$

or

$$\int_a^b \left(\sum_{k=1}^{\infty} u_k(x) \right)\ dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)\ dx,$$

For a proof of this theorem refer to Rudin [Rud53, pg. 121–122].

We saw in Proposition 5.2.8 that the trigonometric series defining $v(r, \theta)$ in (5.143),

$$v(r, \theta) = a_o + \sum_{n=0}^{\infty} r^n [a_n \cos n\theta + b_n \sin n\theta],\ 0 \leqslant r \leqslant 1,\ -\pi \leqslant \theta \leqslant \pi, \quad (5.159)$$

converges in D_1, provided that $g \in L^1(-\pi, \pi)$, or absolutely integrable on $[-\pi, \pi]$. In the next proposition we will use the Weierstrass M–Test for uniform convergence, (see [Rud53, Theorem 7.10, pg. 119]), in order to show that, for the case in which g is piece–wise C^1 and 2π–periodic, then the series in (5.159), where the a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of g, converges uniformly in $\overline{D}_1$, the closed unit disk in $\mathbb{R}^2$.

Proposition 5.2.13 (Uniform Convergence of Series in (5.143))**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.148) and (5.150), and b_n, for $n = 1, 2, 3, \ldots$ be as in (5.151), where g is a piecewise C^1, 2π–periodic function. Then, the series defining v in (5.159) converges uniformly in $\overline{D}_1$.

Proof: The assumptions that g is piece–wise C^1 and 2π periodic imply that the Fourier coefficients of g satisfy the estimate in (5.158); namely,

$$\sum_{k=0}^{\infty} (|a_k| + |b_k|) < \infty. \quad (5.160)$$

See [Tol62, pp. 80-81] for details of the calculations leading up to (5.160).

Next, us the triangle inequality to estimate the absolute values of the terms of the series in (5.159) to get

$$|r^n [a_n \cos(n\theta) + b_n \sin(n\theta)]| \leqslant |a_n| + |b_n|, \quad \text{for all } n = 1, 2, 3, \ldots,$$

and all $r \in [0, 1]$ and $\theta \in [-\pi, \pi]$. Thus, the absolute values of the terms of the series in (5.159) are "majorized" by the terms of the convergent series in

(5.160). It then follows by the the Weierstrass M–Test for uniform convergence ([Rud53, Theorem 7.10, pg. 119]) that the series in (5.159) converges uniformly for $r \in [0,1]$ and $\theta \in [-\pi,\pi]$. ■

We will get a chance to use Weierstrass M–Test for uniform convergence once again to justify the following calculations based on the trigonometric series representation for $v(r,\theta)$ in (5.159) and the assumption that g is a piecewise C^1, 2π–periodic function.

First, substitute the formulas defining the Fourier coefficients of g in (5.148), (5.150) and (5.151) into the right–hand side (5.159) to get

$$\begin{aligned}
v(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\ d\xi + \sum_{n=0}^{\infty} r^n \left[\frac{1}{\pi}\left(\int_{-\pi}^{\pi} g(\xi)\cos(n\xi)\ d\xi\right)\cos(n\theta)\right. \\
&\qquad \left. + \frac{1}{\pi}\left(\int_{-\pi}^{\pi} g(\xi)\sin(n\xi)\ d\xi\right)\sin(n\theta)\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\ d\xi + \frac{1}{\pi}\sum_{n=0}^{\infty} r^n \left[\int_{-\pi}^{\pi} \cos(n\theta)\cos(n\xi)g(\xi)\ d\xi\right. \\
&\qquad \left. + \int_{-\pi}^{\pi} \sin(n\theta)\sin(n\xi)g(\xi)\ d\xi\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\ d\xi \\
&\qquad + \frac{1}{\pi}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} r^n[\cos(n\theta)\cos(n\xi)+\sin(n\theta)\sin(n\xi)]g(\xi)\ d\xi,
\end{aligned}$$

which can be written as

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\ d\xi + \frac{1}{\pi}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} r^n\cos[n(\theta-\xi)]g(\xi)\ d\xi, \qquad (5.161)$$

for $0 \leqslant r < 1$ and $\theta \in [-\pi,\pi]$, by virtue of the trigonometric identity

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Next, we will interchange the order of integration and summation in (5.161). This is justified by the fact that the series

$$\sum_{n=1}^{\infty} r^n\cos(n\xi)$$

converges uniformly in $\xi \in [-\pi,\pi]$ for $0 \leqslant r < 1$. To see why this is the case, note that

$$|r^n\cos(n\xi)| \leqslant r^n, \qquad \text{for all } n = 1,2,3,\ldots$$

Thus, the assertion follows by the Weierstrass M–Test for uniform convergence, for $0 \leqslant r < 1$.

Hence, interchanging the order of summation and integration in (5.161), we can write

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + \sum_{n=1}^{\infty} 2r^n \cos[n(\theta-\xi)]\right] g(\xi)\ d\xi, \tag{5.162}$$

for $0 \leqslant r < 1$ and $\theta \in [-\pi,\pi]$. Putting

$$P(r,\theta) = \frac{1}{2\pi}\left[1 + \sum_{n=1}^{\infty} 2r^n \cos(n\theta)\right], \quad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi,\pi], \tag{5.163}$$

we see that (5.164) can be written as

$$v(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\xi) g(\xi)\ d\xi, \quad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi,\pi]. \tag{5.164}$$

5.2.4 The Poisson Kernel for the Unit Disk

The function P defined in (5.163) is called the **Poisson kernel** for the unit disk in $\mathbb{R}^2$, and the expression on the right–hand side of (5.164) is called the **Poisson integral representation** for v. In this section and the next, we will prove several important properties of the Poisson kernel and the Poisson integral in (5.164).

We will first show that the series defining the Poisson kernel in (5.163) converges uniformly over $\theta \in [-\pi,\pi]$ for each $0 \leqslant r < 1$. This will justify term–by–term integration of the series. We will use the Weierstrass M–Test for uniform convergence. Thus, we first estimates the absolute values of the of the terms of the series,

$$|2r^n \cos(n\theta)| \leqslant 2r^n, \quad \text{for all } \theta \in [-\pi,\pi]. \tag{5.165}$$

It follows from (5.165) that the absolute values of the terms of the series in (5.163) are "majorized" by the terms of the convergent geometric series

$$\sum_{n=1}^{\infty} 2r^n,$$

for $0 \leqslant r < 1$. Hence, the Weierstrass M–Test applies, and we conclude that the series defining $P(r,\theta)$ in (5.163) converges uniformly in θ for $0 \leqslant r < 1$. This arguments can be carried out further to prove that, for any $0 < R < 1$, the series defining $P(r,\theta)$ in (5.163) converges uniformly for $\theta \in [-\pi,\pi]$ and $r \in [0,R]$. Hence, $P(r,\theta)$ defines a continuous function in the open unit disc, D_1, in $\mathbb{R}^2$. This follows from the following important consequence of the uniform convergence of a sequence of continuous functions:

Proposition 5.2.14 (Uniform Limit of Continuous Functions)**.** Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to a functions $f\colon [a, b] \to \mathbb{R}$. Then, f is continuous.

For a proof of this proposition see Rudin [Rud53, Theorem 7.12, pg. 20].

We will next show that, in fact, the Poisson kernel is C^2 in D_1 and that it solves Laplace's equation in D_1. In order to show that the partial derivatives of P exist, we need to show that the series

$$\sum_{n=1}^{\infty} 2nr^{n-1}\cos(n\theta) \quad \text{and} \quad \sum_{n=1}^{\infty} 2nr^{n}\sin(n\theta) \tag{5.166}$$

converge uniformly. This assertion will follow from the following proposition

Proposition 5.2.15 (Term–by–Term Differentiation)**.** Let (u_k) be a sequence of functions that are differentiable over a closed and bounded interval, $[a, b]$. Assume that the series

$$\sum_{k=1}^{\infty} u_k'$$

converges uniformly over $[a, b]$. Assume also that the series

$$\sum_{k=1}^{\infty} u_k(x_o)$$

converges at some point x_o in $[a, b]$. Then, the series converges

$$\sum_{k=1}^{\infty} u_k$$

converges uniformly to function f that is differentiable over $[a, b]$, and

$$f'(x) = \sum_{k=1}^{\infty} u_k'(x), \qquad \text{for all } x \in [a, b];$$

or

$$\frac{d}{dx}\left[\sum_{k=1}^{\infty} u_k(x)\right] = \sum_{k=1}^{\infty} u_k'(x) \ ,$$

for all $x \in [a, b]$.

This proposition can be proved by applying Theorem 7.17 in [Rud53, pg. 124].

In order to see that the series in (5.166) converge absolutely and uniformly, first note that

$$|2nr^{n-1}\cos(n\theta)| \leqslant 2nr^{n-1} \quad \text{and} \quad |2nr^{n}\sin(n\theta)| \leqslant 2nr^2$$

for all $\theta \in [-\pi, \pi]$; so that the series in (5.166) are "majorized" by the series

$$\sum_{n=1}^{\infty} 2nr^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} 2nr^{n}, \tag{5.167}$$

respectively; both of the series in (5.167) converge by the Root Test (or the Ratio Test), since $0 \leqslant< 1$. It then follows by the Weierstrass M–Test for uniform convergence, that the series in (5.167) converge uniformly in θ. The same argument applied to $r \in [0, R]$, where where $R < 1$, yields that the series in (5.166) are absolutely and uniformly convergent for $\theta\imath[-\pi, \pi]$ and $r \in [0, R]$. This time the series in (5.166) are "majorized" by the convergent series

$$\sum_{n=1}^{\infty} 2nR^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} 2nR^{n}.$$

It the follows from Proposition 5.2.15 that the partial derivatives of the Poisson kernel in (5.163) have partial derivatives in D_1 given by

$$\frac{\partial}{\partial r}[P(r,\theta)] = \frac{1}{2\pi}\sum_{n=1}^{\infty} 2nr^{n-1}\cos(n\theta), \quad 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.168}$$

and

$$\frac{\partial}{\partial \theta}[P(r,\theta)] = -\frac{1}{2\pi}\sum_{n=1}^{\infty} 2nr^{n}\sin(n\theta), \quad 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.169}$$

where we have differentiated the series in (5.163) term–by–term. A similar argument can be used to obtain the second partial derivatives of the of the Poisson kernel:

$$\frac{\partial^2}{\partial r^2}[P(r,\theta)] = \frac{1}{2\pi}\sum_{n=1}^{\infty} 2n(n-1)r^{n-2}\cos(n\theta), \quad 0 \leqslant r < 1,\ \theta \in [-\pi, \pi], \tag{5.170}$$

and

$$\frac{\partial^2}{\partial \theta^2}[P(r,\theta)] = -\frac{1}{2\pi}\sum_{n=1}^{\infty} 2n^2r^{n}\cos(n\theta), \quad 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.171}$$

where the series in (5.168) and (5.169) have been differentiated term–by–term.

Next, substitute the partial derivatives in (5.168), (5.170) and (5.171) into the expression for the Laplacian of P in polar coordinates to get

$$\begin{aligned}\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}+\frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} &= \frac{1}{2\pi}\sum_{n=1}^{\infty}2n(n-1)r^{n-2}\cos(n\theta)\\ &\quad +\frac{1}{2\pi r}\sum_{n=1}^{\infty}2nr^{n-1}\cos(n\theta)\\ &\quad -\frac{1}{2\pi r^2}\sum_{n=1}^{\infty}2n^2r^n\cos(n\theta) \qquad (5.172)\\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}[n(n-1)+n-n^2]r^{n-2}\cos(n\theta)\\ &= 0,\end{aligned}$$

for all $\theta\in[-\pi,\pi]$ and $0\leqslant r<1$. We have therefore shown that the Poisson kernel solves Laplace's equation in the open unit disc.

Next, integrating the series in (5.163) over the interval $[-\pi,\pi]$, which is justified by the uniform convergence of the series, to obtain

$$\int_{-\pi}^{\pi}P(r,\theta)\;d\theta=1,\qquad \text{for all } 0\leqslant r<1. \tag{5.173}$$

The series defining the Poisson kernel in (5.163) can actually be evaluated by using the identity

$$2\cos(n\theta)=e^{in\theta}+e^{-in\theta},$$

and then adding geometric series. Indeed,

$$\begin{aligned}\sum_{n=1}^{\infty}2r^n\cos(n\theta) &= \sum_{n=1}^{\infty}r^n[e^{in\theta}+e^{-in\theta}]\\ &= \sum_{n=1}^{\infty}r^n[e^{i\theta}]^n+\sum_{n=1}^{\infty}r^n[e^{-i\theta}]^n\\ &= \sum_{n=1}^{\infty}[re^{i\theta}]^n+\sum_{n=1}^{\infty}[re^{-i\theta}]^n,\end{aligned}$$

so that, since $|re^{\pm i\theta}|=r<1$, for all θ,

$$\sum_{n=1}^{\infty}2r^n\cos(n\theta) = \frac{re^{i\theta}}{1-re^{i\theta}}+\frac{re^{-i\theta}}{1-re^{-i\theta}},$$

which simplifies to

$$\sum_{n=1}^{\infty}2r^n\cos(n\theta) = \frac{re^{i\theta}-r^2+re^{-i\theta}-r^2}{1-re^{i\theta}-re^{-i\theta}+r^2},$$

or

$$\sum_{n=1}^{\infty} 2r^n \cos(n\theta) \quad = \quad \frac{r[e^{i\theta}+e^{-i\theta}]-2r^2}{1-r[e^{i\theta}+e^{-i\theta}]+r^2},$$

or

$$\sum_{n=1}^{\infty} 2r^n \cos(n\theta) \quad = \quad \frac{2r\cos(\theta)-2r^2}{1-2r\cos(\theta)+r^2}, \quad 0 \leqslant r < 1, \theta \in [-\pi,\pi]. \qquad (5.174)$$

Substituting the value of the series in (5.174) into (5.163) then yields the formula

$$P(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos(\theta)+r^2}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi,\pi], \qquad (5.175)$$

for the Poisson kernel.

We will next use the formula in (5.175) for the Poisson kernel for the unit disk to derive further properties of the Poisson kernel. We summarize these properties, as well as the ones we have already established using the representation in (5.163) in the following proposition.

Proposition 5.2.16 (Properties of the Poisson Kernel)**.** Let $P(r,\theta)$ be given by (5.175), or its equivalent representation as an infinite series in (5.163). Then, the function $P\colon [0,1)\times[-\pi,\pi] \to \mathbb{R}$ satisfies the following:

(i) $P(r,\theta) > 0$ for all $(r,\theta) \in [0,1)\times[-\pi,\pi]$;

(ii) $P \in C^\infty([0,1)\times[-\pi,\pi])$;

(iii) P is harmonic in D_1;

(iv) $\displaystyle\int_{-\pi}^{\pi} P(r,\theta-\xi)\, d\xi = 1$, for all $\xi \in \mathbb{R}$ and all $0 \leqslant r < 1$.

(v) $\displaystyle\lim_{r\to 1^-} P(r,\theta-\xi) = 0$, for $\xi \neq \theta$ and $|\xi-\theta| < \pi$;

(vi) $\displaystyle\lim_{r\to 1^-} P(r,\theta-\xi) = +\infty$, for $\xi = \theta$.

Proof: In order to prove (i) and (ii), first note that, for all $\theta \in \mathbb{R}$ and $r \geqslant 0$,

$$2r\cos\theta \leqslant 2r,$$

so that

$$1-2r\cos\theta+r^2 \geqslant 1-2r+r^2,$$

or

$$1-2r\cos(\theta)+r^2 \geqslant (1-r)^2, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in \mathbb{R}. \qquad (5.176)$$

It follows from (5.176) and the formula for $P(r,\theta)$ in (5.175) that $P(r,\theta)$ is defined for all $r \in [0,r)$ and all $\theta \in \mathbb{R}$, and $P(r,\theta) > 0$ for $r \in [0,r)$ and all $\theta \in \mathbb{R}$; we have therefore establlished (i).

From (5.176) we also obtain that

$$1 - 2r\cos(\theta) + r^2 > 0, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in \mathbb{R}.$$

Thus, the denominator in the formula for $P(r, \theta)$ in (5.175) is not zero for $0 \leqslant r < 1$ and $\theta \in \mathbb{R}$; hence, the since the numerator and denominator of the expression defining $P(r, \theta)$ in (5.175) are C^∞ functions, (ii) also follows.

We have already established that P satisfies Laplace's equation in D_1 (see the calculations leading up to (5.172) on page 113) using the definition of P in (5.163). Thus, P is harmonic in D_1 and so we have established (iii).

The integral identity in (iv) will follow from (5.173) and the 2π–periodicity of $P(r, \theta)$ in θ. Indeed, making the change of variables $\zeta = \theta - \xi$ in the integral in (iv) we have

$$\begin{aligned}
\int_{-\pi}^{\pi} P(r, \theta - \xi)\ d\xi &= -\int_{\theta+\pi}^{\theta-\pi} P(r, \zeta)\ d\zeta \\
&= \int_{\theta-\pi}^{\theta+\pi} P(r, \zeta)\ d\zeta \\
&= \int_{-\pi}^{\pi} P(r, \zeta)\ d\zeta \\
&= 1,
\end{aligned}$$

for all $\theta \in \mathbb{R}$.

Next, use the formula for $P(r, \theta)$ in (5.175) to obtain that

$$P(r, \theta - \xi) = \frac{1}{2\pi}\ \frac{1 - r^2}{1 - 2r + r^2},$$

for $\theta = \xi$, from which we get that

$$P(r, \theta - \xi) = \frac{1}{2\pi}\ \frac{1 + r}{1 - r}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta = \xi. \tag{5.177}$$

The assertion in (vi) follows from (5.177).

To prove (v), first note that

$$\begin{aligned}
\lim_{r \to 1^-} [1 - 2r\cos(\theta - \xi) + r^2] &= 2 - 2\cos(\theta - \xi) \\
&= \sin^2(\theta - \xi)
\end{aligned}$$

so that

$$\lim_{r \to 1^-} [1 - 2r\cos(\theta - \xi) + r^2] \neq 0, \qquad \text{for } \xi \neq 0 \text{ and } |\xi - \theta| < \pi \tag{5.178}$$

The assertion in (v) then follows from (5.178) and the expression for the Poisson kernel in (5.175). ■

5.2.5 The Poisson Integral Representation

Let $g\colon [-\pi,\pi]\to\mathbb{R}$ be a continuous function that can be extended to a continuous 2π–periodic function in $\mathbb{R}$. We then have that

$$|g(\theta)|\leqslant M, \quad \text{for all } \theta\in[-\pi,\pi], \tag{5.179}$$

and some positive constant M. The goal of this section is to use the properties of the Poisson kernel listed in Proposition 5.2.16 to prove that the function $u\colon \overline{D}_1\to\mathbb{R}$ defined by

$$u(r,\theta)=\begin{cases}\displaystyle\int_{-\pi}^{\pi} P(r,\theta-\xi)g(\xi)\;d\xi, & \text{for } 0\leqslant r<1,\ \theta\in[-\pi,\pi];\\[2ex] g(\theta), & \text{for } r=1,\ \theta\in[-\pi,\pi],\end{cases} \tag{5.180}$$

where $P(r,\theta)$ denotes the Poisson kernel for the unit disk in $\mathbb{R}^2$ given in (5.163) or (5.175), solves the Dirichlet problem for the unit disk in $\mathbb{R}^2$.

We first show that $u\in C^2(D_1)$ and that it solves Laplace's equation in D_1; in polar coordinates,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0, \quad \text{for } 0\leqslant r<1,\ \theta\in[-\pi,\pi]. \tag{5.181}$$

This will follow from (iii) in Proposition 5.2.16, provided we can show that differentiation under the integral sign in the first part of the definition of u in (5.180) is valid. Indeed, property (iii) in Proposition 5.181 says that

$$\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}+\frac{1}{r^2}\frac{\partial^2 P}{\partial\theta^2}=0, \quad \text{for } 0\leqslant r<1,\ \theta\in[-\pi,\pi]. \tag{5.182}$$

Thus, assuming for the moment that differentiation under the integral sign in (5.180) is valid, we have that, for $0\leqslant r<1$ and $\theta\in[-\pi,\pi]$,

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \;=\; & \int_{-\pi}^{\pi}\frac{\partial^2}{\partial r^2}[P(r,\theta-\xi)]g(\xi)\;d\xi\\ & +\int_{-\pi}^{\pi}\frac{1}{r}\frac{\partial}{\partial r}[P(r,\theta-\xi)]g(\xi)\;d\xi\\ & +\int_{-\pi}^{\pi}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]g(\xi)\;d\xi,\end{aligned}$$

which can be written as

$$\Delta u=\int_{-\pi}^{\pi}\left[\frac{\partial^2}{\partial r^2}[P(r,\theta-\xi)]+\frac{1}{r}\frac{\partial}{\partial r}[P(r,\theta-\xi)]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]\right]g(\xi)\;d\xi,$$

where we have used the short–hand notation, Δu, for the Laplacian of u. The fact that u is harmonic in D_1 then follows from the previous identity and (5.182).

We will next see that differentiation under the integral sign is justified. In order to do this, we first note that the continuity of g implies that there exists a positive constant, M, such that

$$|g(\theta)| \leqslant M, \qquad \text{for all } \theta \in [-\pi, \pi]. \tag{5.183}$$

In view of (5.183) and (5.182), in order to justify the differentiation under the integral sign in the first part of the definition of u in (5.180), it suffices to prove that

$$\frac{\partial}{\partial\theta}[P(r, \theta - \xi)], \quad \frac{\partial}{\partial r}[P(r, \theta - \xi)] \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}[P(r, \theta - \xi)]$$

are absolutely integrable over $[-\pi, \pi]$ for each $[-\pi, \pi]$.

Use (5.175) to compute

$$\frac{\partial}{\partial\theta}[P(r, \theta - \xi)] = -\frac{1}{2\pi}\ \frac{(1 - r^2)2r\sin(\theta - \xi)}{(1 - 2r\cos(\theta - \xi) + r^2)^2},$$

which can be written as

$$\frac{\partial}{\partial\theta}[P(r, \theta - \xi)] = -\frac{2r\sin(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} P(r, \theta - \xi), \tag{5.184}$$

by virtue of the expression for the Poisson kernel in (5.175). Next, take absolute values on both sides of (5.184) and use the estimate in (5.176) to get

$$\left|\frac{\partial}{\partial\theta}[P(r, \theta - \xi)]\right| \leqslant \frac{2r}{(1 - r)^2}\ P(r, \theta - \xi), \tag{5.185}$$

where we have used the positivity of the Poisson kernel in (i) of Proposition 5.2.16. Integrating on both sides of the inequality in (5.185) form $-\pi$ to π and using property (iv) in Proposition 5.2.16 we obtain that

$$\int_{-\pi}^{\pi} \left|\frac{\partial}{\partial\theta}[P(r, \theta - \xi)]\right| d\xi \leqslant \frac{2r}{(1 - r)^2}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi],$$

which shows that $\dfrac{\partial}{\partial\theta}[P(r, \theta - \xi)]$ is absolutely integrable over $[-\pi, \pi]$ for $0 \leqslant r < 1$ and $\theta \in [-\pi, \pi]$.

Next, take partial derivative with respect to θ on both sides of (5.184) to get

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}[P(r, \theta - \xi)] \quad = \quad & -\frac{2r\cos(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} P(r, \theta - \xi) \\ & + \frac{4r^2\sin^2(\theta - \xi)}{(1 - 2r\cos(\theta - \xi) + r^2)^2} P(r, \theta - \xi) \\ & - \frac{2r\sin(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2}\frac{\partial}{\partial\theta}[P(r, \theta - \xi)],\end{aligned}$$

so that, in view of (5.184),

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)] &= -\frac{2r\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}P(r,\theta-\xi)\\ &\quad+\frac{4r^2\sin^2(\theta-\xi)}{(1-2r\cos(\theta-\xi)+r^2)^2}P(r,\theta-\xi)\\ &\quad+\frac{4r^2\sin^2(\theta-\xi)}{(1-2r\cos(\theta-\xi)+r^2)^2}P(r,\theta-\xi),\end{aligned}$$

or

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)] &= -\frac{2r\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}P(r,\theta-\xi)\\ &\quad+\frac{8r^2\sin^2(\theta-\xi)}{(1-2r\cos(\theta-\xi)+r^2)^2}P(r,\theta-\xi).\end{aligned} \tag{5.186}$$

Taking absolute values on both sides of (5.186) and applying the triangle inequality, we obtain

$$\left|\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]\right| \leqslant \frac{2r}{(1-r)^2}P(r,\theta-\xi)+\frac{8r^2}{(1-r)^4}P(r,\theta-\xi), \tag{5.187}$$

where we have also used the estimate in (5.176) and the positivity of the Poisson kernel (see property (i) in Proposition 5.2.16). Integrating from $-\pi$ to π on both sides of (5.187) then yields

$$\int_{-\pi}^{\pi}\left|\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]\right|\,d\xi \leqslant \frac{2r}{(1-r)^2}+\frac{8r^2}{(1-r)^4}, \qquad \text{for } 0\leqslant r<1,\ \theta\in\mathbb{R},$$

where we have also used property (iv) in Proposition 5.2.16; thus, we have shown that $\dfrac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]$ is absolutely integrable over $[-\pi,\pi]$ for $0\leqslant r<1$ and $\theta\in\mathbb{R}$.

Next, differentiate the Poisson kernel in (5.175) with respect to r, for $0\leqslant r<1$, to obtain

$$\frac{\partial}{\partial r}[P(r,\theta-\xi)] = \frac{1}{2\pi}\,\frac{-2r}{1-2r\cos(\theta-\xi)+r^2}-\frac{1}{2\pi}\,\frac{(1-r^2)(2r-2\cos(\theta-\xi))}{(1-2r\cos(\theta-\xi)+r^2)^2},$$

where we have applied the Product Rule; so that, in view of the expression for the Poison kernel in (5.175),

$$\frac{\partial}{\partial r}[P(r,\theta-\xi)] = -\frac{2r}{1-r^2}P(r,\theta-\xi)-\frac{2r-2\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}P(r,\theta-\xi),$$

or

$$\frac{\partial}{\partial r}[P(r,\theta-\xi)] = \left[\frac{-2r}{1-r^2}+\frac{2r-\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}\right]P(r,\theta-\xi), \tag{5.188}$$

for $0 \leqslant r < 1$ and all ξ and θ in $\mathbb{R}$.

Next, take absolute values on both sides of (5.188), applying the triangle inequality, and use the estimate in (5.176) to obtain

$$\left|\frac{\partial}{\partial r}[P(r,\theta-\xi)]\right| \leqslant \left[\frac{2r}{1-r^2}+\frac{2r+1}{(1-r)^2}\right] P(r,\theta-\xi), \tag{5.189}$$

for $0 \leqslant r < 1$ and $\theta, \xi \in \mathbb{R}$, where we have also used the positivity of the Poisson kernel (see property (i) in Proposition 5.2.16).

Integrating from $-\pi$ to π on both sides of (5.189) and using property (iv) in Proposition 5.2.16 then yields

$$\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial r}[P(r,\theta-\xi)]\right| \, d\xi \leqslant \frac{2r}{1-r^2}+\frac{2r+1}{(1-r)^2}, \qquad \text{for } 0 \leqslant r < 0,\ \theta \in \mathbb{R},$$

which shows that $\dfrac{\partial}{\partial r}[P(r,\theta-\xi)]$ is absolutely integrable over $[-\pi,\pi]$ for $0 \leqslant r < 1$ and $\theta \in [-\pi,\pi]$.

Hence, differentiation under the integral sign in the first part of the definition of u in (5.180) is justified. We have therefore established that the function u defined in (5.180) is in $C^2(D_1)$ and satisfies Laplace's equation. It remains to prove that $u \in C(\overline{D}_1)$ and that it satisfies the boundary conditions in problem (5.100). This will be accomplished once we prove the following lemma:

Lemma 5.2.17 (Boundary Limits of the Poisson Integral Representation)**.** Let u be as given in (5.180) where g is continuous on $[-\pi,\pi]$. Then, for every $\zeta \in [-pi,\pi]$,

$$\lim_{(r,\theta)\to(1,\zeta)} |u(r,\theta)-g(\zeta)| = 0. \tag{5.190}$$

Proof: First consider the case in which $\zeta \in (-\pi,\pi)$.

Let $\varepsilon > 0$ be given. Since g is is continuous on $[-\pi,\pi]$, there exists $\delta_1 > 0$ such that $\delta_1 < \dfrac{\pi}{2}$, and

$$|\xi-\zeta| < \delta_1 \Rightarrow \xi \in (-\pi,\pi) \text{ and } |g(\xi)-g(\zeta)| < \frac{\varepsilon}{3}. \tag{5.191}$$

Next, use property (iii) of the Poisson kernel in Proposition 5.2.16 to write

$$\begin{aligned} u(r,\theta)-g(\zeta) &= \int_{-\pi}^{\pi} P(r,\theta-\xi)g(\xi)\, d\xi - g(\zeta)\int_{-\pi}^{\pi} P(r,\theta-\xi)\, d\xi \\ &= \int_{-\pi}^{\pi} P(r,\theta-\xi)g(\xi)\, d\xi - \int_{-\pi}^{\pi} P(r,\theta-\xi)g(\zeta)\, d\xi, \end{aligned}$$

so that

$$u(r,\theta)-g(\zeta) = \int_{-\pi}^{\pi} P(r,\theta-\xi)[g(\xi)-g(\zeta)]\, d\xi. \tag{5.192}$$

Next, take absolute values on both sides of (5.192) and use the positivity of the Poisson kernel (see property (i) in Proposition 5.2.16) to obtain that

$$|u(r,\theta) - g(\zeta)| \leqslant \int_{-\pi}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi. \tag{5.193}$$

We'll next divide the integral on the right–hand–side of (5.193) into three integrals over the domains $[-\pi, \zeta-\delta_1]$, $[\zeta-\delta_1, \zeta+\delta_1]$ and $[\zeta+\delta_1, \pi]$, respectively. We first estimate the integral over $[\zeta-\delta_1, \zeta+\delta_1]$ using (5.192) to get

$$\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < \frac{\varepsilon}{3}\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)\ d\xi,$$

so that, by virtue of the positivity of the Poisson (property (i) in Proposition 5.2.16)

$$\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < \frac{\varepsilon}{3}\int_{-\pi}^{\pi} P(r,\theta-\xi)\ d\xi;$$

hence, by property (iv) in Proposition 5.2.16,

$$\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < \frac{\varepsilon}{3}. \tag{5.194}$$

Next, we estimate the integral over $[\zeta+\delta_1, \pi]$. Using the estimate in (5.183) and the triangle inequality we obtain

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < 2M\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)\ d\xi; \tag{5.195}$$

Then, for

$$|\theta-\zeta| < \frac{\delta_1}{2}, \tag{5.196}$$

we obtain from (5.195) and the positivity of the Poisson kernel that

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < 2M\int_{\theta+\delta_1/2}^{\pi} P(r,\theta-\xi)\ d\xi; \tag{5.197}$$

Thus, making the change of variables $\omega = \xi - \theta$ in (5.197) we get

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < 2M\int_{\delta_1/2}^{\pi-\theta} P(r,\theta-\xi)\ d\xi,$$

so that, by the positivity of the Poisson kernel,

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < 2M\int_{\delta_1/2}^{\pi} P(r,\omega)\ d\omega. \tag{5.198}$$

Now, it follows from the properties of the cosine function that

$$\frac{\delta_1}{2} < \omega < \pi \Rightarrow \cos(\omega) < \cos(\delta_1/2),$$

so that, for $0 < r < 1$,

$$\frac{\delta_1}{2} < \omega < \pi \Rightarrow 1 - 2r\cos(\omega) + r^2 > 1 - 2r\cos(\delta_1/2) + r^2;$$

so that, by the expression for the Poisson kernel in

$$P(r,\omega) < P(r,\delta_1/2), \quad \text{for all } \omega \in [\delta_1/2, \pi]. \tag{5.199}$$

It then follows from (5.198) and (5.199) that

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < 2M \int_{\delta_1/2}^{\pi} P(r,\delta_1/2)\ d\omega,$$

so that

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < 2M\left(\pi - \frac{\delta_1}{2}\right) P(r,\delta_1/2). \tag{5.200}$$

Now, it follows property (v) in Proposition 5.2.16 that there exists $\delta_2 > 0$ such that

$$|r - 1| < \delta_2 \Rightarrow P(r,\delta_1/2) < \frac{\varepsilon}{3M(2\pi - \delta_1)} \tag{5.201}$$

Thus, combining (5.200), (5.196) and (5.201) we see that

$$|r-1| < \delta_2 \text{ and } |\theta - \zeta| < \frac{\delta_1}{2} \Rightarrow \int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < \frac{\varepsilon}{3}. \tag{5.202}$$

Similar calculations similar to those leading to (5.202) show that there exists $\delta_3 > 0$ such that

$$|r-1| < \delta_3 \text{ and } |\theta - \zeta| < \frac{\delta_1}{2} \Rightarrow \int_{-\pi}^{\zeta-\delta_1} P(r,\theta-\xi)|g(\xi) - g(\zeta)|\ d\xi < \frac{\varepsilon}{3}. \tag{5.203}$$

Letting $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2, \delta_3\right\}$, we see that in view of (5.193), (5.194), (5.202) and (5.203), that

$$|r-1| < \delta \text{ and } |\theta - \zeta| < \delta \Rightarrow |u(r,\theta) - g(\zeta)| < \varepsilon.$$

This completes the proof of the boundary limits lemma for the case $\zeta \in (-\pi, \pi)$. The case in which ζ is one of the end–points of the interval $(-\pi, \pi)$ can be treated in an analogous manner to the interior point case using one–sided limits at those points. ■

Appendix A

Differentiating Under the Integral Sign

In this appendix we preset the following result about differentiation under the integral sign.

Proposition A.0.18 (Differentiation Under the Integral Sign)**.** Suppose that $H\colon \mathbb{R}\times\mathbb{R}\times(a,b)\to\mathbb{R}$ is a C^1 function. Define

$$h(x,t)=\int_a^t H(x,t,s)\ ds, \qquad \text{for all } x\in\mathbb{R},\ t\in\mathbb{R}.$$

Assume that the functions H, $\dfrac{\partial}{\partial x}[H(x,t,s)]$ and $\dfrac{\partial}{\partial t}[H(x,t,s)]$ are absolutely integrable over (a,b). Then, the h is C^1 and its partial derivatives are given by

$$\frac{\partial}{\partial x}[h(x,t)]=\int_a^t \frac{\partial}{\partial x}[H(x,t,s)]\ ds$$

and

$$\frac{\partial}{\partial t}[h(x,t)]=H(x,t,t)+\int_a^t \frac{\partial}{\partial t}[H(x,t,s)]\ ds.$$

Proposition A.0.18 can be viewed as a generalization of the Fundamental Theorem of Calculus and is a special case of Leibnitz Rule.

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