## Assignment #14

**Due on Wednesday, April 8, 2015**

**Read** Section 6.1 on *Nondimensionalization* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

**Read** Section 5.1, on *Equilibrium Point Analysis,* in Blanchard, Devaney and Hall.

**Read** Section 5.2, on *Qualitative Analysis,* in Blanchard, Devaney and Hall.

**Background and Definitions**.
In Section 6.1 in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`, we derived the following system of ordinary differential equations for the chemostat system,

$$\begin{cases} \dfrac{dn}{dt} = \dfrac{bnc}{a+c} - \dfrac{F}{V}n; \\[2ex] \dfrac{dc}{dt} = \dfrac{F}{V}c_o - \dfrac{F}{V}c - \dfrac{\alpha bnc}{a+c}. \end{cases} \tag{1}$$

The variables $n$ and $c$ are the bacterial population density and nutrient concentration, respectively, in the chemostat; these are assumed to be differentiable functions of time, $t$. The parameter $c_o$, $F$, $V$, $\alpha$, $a$ and $b$ have the following interpretations:

- $c_o$ is the nutrient concentration in a reservoir that feeds the chemostat chamber at a constant rate $F$;
- $F$ is also the rate at which culture is drawn from the chemostat chamber;
- $V$ is the fixed volume of the culture;
- $\alpha$ is related to the yield, $Y = 1/\alpha$, which is the number of new cells produced in the chemostat due to consumption of one unit of nutrient;
- $b$ is the maximum *per–capita* growth rate allowed by the medium, and $a$ is the nutrient concentration at which the *per–capita* growth rate is $b/2$.

1. Introduce new dimensionless variables

$$\widehat{n} = \frac{n}{\mu}, \quad \widehat{c} = \frac{c}{a}, \quad \text{and} \quad \tau = \frac{t}{\lambda}, \tag{2}$$

where $\mu$ and $\lambda$ are scaling parameters having units of cells/volume and time, respectively.

Verify that the second equation in the system in (1) can be written in the form

$$\frac{d\widehat{c}}{d\tau} = \alpha_2 - \frac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{c},$$

where

$$\alpha_2 = \frac{c_o}{a} \tag{3}$$

and

$$\mu = \frac{a}{\alpha b \lambda}. \tag{4}$$

2. Verify that the parameter $\alpha_2$ in (3) is dimensionless and that the units of $\mu$ defined in (4) are indeed cells/volume. Justify your answers.

3. In Section 6.1 in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`, the system in (1) was nondimensionalized to yield the system

$$\begin{cases} \dfrac{d\widehat{n}}{d\tau} &= \alpha_1 \dfrac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{n}; \\[2ex] \dfrac{d\widehat{c}}{d\tau} &= \alpha_2 - \dfrac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{c}. \end{cases} \tag{5}$$

   (a) Compute the equilibrium solutions of the system in (5) in the $\widehat{n}\widehat{c}$–phase plane.

   (b) Give interpretations for each of the equilibrium points obtained in part (a). Give conditions under which the system in (5) yields biologically feasible equilibrium solutions.

4. Put

$$F(\widehat{n},\widehat{c}) = \begin{pmatrix} \alpha_1 \dfrac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{n} \\[2ex] \alpha_2 - \dfrac{\widehat{n}\widehat{c}}{1+\widehat{c}} - \widehat{c} \end{pmatrix}$$

   Compute the derivative matrix, $DF(\widehat{n},\widehat{c})$, of $F$.

5. Compute the eigenvalues of $DF(\widehat{n},\widehat{c})$ at the equilibrium points found in Problem 3 and use this information to determine their stability properties. What do you conclude about the chemostat system?