Assignment #18

Due on Friday, April 24, 2015

Read Section 6.4, on *Analysis of the Pendulum Equation,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.3, on *Hamiltonian Systems,* in Blanchard, Devaney and Hall.

Read Section 5.4, on *Dissipative Systems,* in Blanchard, Devaney and Hall.

Background and Definitions.

Lyapunov Functions. Suppose that f and g are continuous functions with continuous partial derivatives defined in some domain, D, of $\mathbb{R}^2$. A differentiable function $V\colon D \to \mathbb{R}$ is said to be a Lyapunov function of the system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\ \\ \dfrac{dy}{dt} = g(x,y), \end{cases} \tag{1}$$

if, for any solutions curve $(x(t), y(t))$ of (1) that is not an equilibrium point of (1),

$$\frac{d}{dt}[V(x(t), y(t)] \leqslant 0, \qquad \text{for all } t \in \mathbb{R}.$$

Gradient Systems. Let F be a real–valued, derivatives function defined in some domain, D, of $\mathbb{R}^2$. The system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \nabla F(x,y)$$

is called a gradient system.

1. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $F(x,y) = x^2 - y^2$, for all $(x,y) \in \mathbb{R}^2$.

   (a) Write down the gradient system associated with the function F.

   (b) Find all equilibrium points of the system obtained in part (a) and determine the nature of their stability.

   (c) Sketch the graph of the function F and sketch its level sets.

   (d) Sketch the phase portrait of the system obtained in part (a).

2. Consider the system

$$\begin{cases} \dot{x} = y; \\ \\ \dot{y} = -x - \dfrac{y}{4} + x^2. \end{cases} \tag{2}$$

Let $V\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$V(x,y) = \frac{1}{2}(x^2+y^2) - \frac{x^3}{3}, \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{3}$$

   (a) Verify that V given in (3) is a Lyapunov function for the system (2).

   (b) Sketch the level sets of V given in (3)

   (c) Sketch the phase portrait of the system in (2) and compare this sketch with the sketch in part (b).

3. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $F(x,y) = x^3 - 3xy^2$, for all $(x,y) \in \mathbb{R}^2$.

   (a) Write down the gradient system associated with the function F.

   (b) Sketch the level sets of F.

   (c) Sketch the phase portrait of the system obtained in part (a).

4. Consider the system

$$\begin{cases} \dot{x} = -x^3; \\ \dot{y} = -y^3. \end{cases} \tag{4}$$

Let $V\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$V(x,y) = \frac{1}{2}(x^2+y^2), \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{5}$$

   (a) Verify that V given in (5) is a Lyapunov function for the system (4).

   (b) Sketch the level sets of V given in (5)

   (c) Sketch the phase portrait of the system in (4) and compare this sketch with the sketch in part (b).

5. For the system $\begin{cases} \dot{x} = x - x^3; \\ \dot{y} = -y, \end{cases}$ sketch nullclines and find all equilibrium points; apply the Principle of Linearized Stability (when applicable) to determine the nature of the equilibrium points; sketch the phase portrait.