## Assignment #7

**Due on Wednesday, February 25, 2015**

**Read** Chapter 3 on *Solving Linear Systems* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

**Read** Section 2.2, on *The Geometry of Systems,* in Blanchard, Devaney and Hall.

**Background and Definitions.**
**Some Facts from the Arithmetic of Complex Numbers.**
The object $z = a + bi$, where $a$ and $b$ are real numbers, denotes a complex number; we write $z \in \mathbb{C}$. The complex number $i$ has the property that $i^2 = -1$.

- **Real Part of a Complex Number**. The real part of $z = a + bi$, denoted by $\text{Re}(z)$, is defined by
$$\text{Re}(z) = a.$$

- **Imaginary Part of a Complex Number**. The imaginary part of $z = a + bi$, denoted by $\text{Im}(z)$, is defined by
$$\text{Im}(z) = b.$$

- **Conjugate of a Complex Number**. The conjugate of $z = a + bi$, denoted by $\overline{z}$, is defined by
$$\overline{z} = a - bi.$$

  **Important Observations**:

  $z \in \mathbb{C}$ is a real number if and only if $\overline{z} = z$.

  $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\text{Im}(z) = \dfrac{z - \overline{z}}{2i}$.

- **Modulus of a Complex Number**. The modulus of $z = a + bi$, denoted by $|z|$, is defined by
$$|z| = \sqrt{a^2 + b^2}.$$

  Note that $z\overline{z} = |z|^2$.

- **Argument of a Complex Number**. The argument of $z = a + bi$, denoted by $\arg(z)$, is defined by
$$\arg(z) = \arctan\left(\frac{b}{a}\right).$$

**Do** the following problems

1. Let $A$ denote a $2 \times 2$ matrix with real entries. Assume that $\lambda \in \mathbb{C}$ is a complex eigenvalue of $A$ with (complex) eigenvector $w$; that is, $w = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1, z_2 \in \mathbb{C}$. Show that $\overline{\lambda}$ is also an eigenvalue of $A$ with corresponding eigenvector $\overline{w} = \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}$.

2. Let $A$ denote a $2 \times 2$ matrix with real entries. Assume that $\lambda = \alpha + i\beta$ is a complex eigenvalue of $A$, where $\beta \neq 0$. Show that the characteristic polynomial of $A$ is given by $p_{_A}(\lambda) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2$.

3. Let $A$ denote a $2 \times 2$ matrix with real entries with eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, with $\beta \neq 0$. Let $w_1 \in \mathbb{C}^2$ be an eigenvector corresponding to $\lambda_1$.

   (a) Show that $w_2 = \overline{w_1}$ is an eigenvector corresponding to $\lambda_2$.

   (b) Define vectors $v_1, v_2 \in \mathbb{R}^2$ by
   $$v_1 = \text{Im}(w_1) = \frac{1}{2i}(w_1 - w_2) \quad \text{and } v_2 = \text{Re}(w_1) = \frac{1}{2}(w_1 + w_2).$$
   Show that the set $\{v_1, v_2\}$ is linearly independent.

   (c) Verify that
   $$\begin{array}{rcl} Av_1 & = & \alpha v_1 + \beta v_2 \\ Av_2 & = & -\beta v_1 + \alpha v_2 \end{array} \tag{1}$$

4. Let $A$, $\lambda_1 = \alpha + i\beta$, $v_1$ and $v_2$ be as in Problem 3.

   (a) Use the result in (1) to deduce that the matrix representation of the linear transformation induced by $A$ relative to the basis $\mathcal{B} = \{v_1, v_2\}$ is
   $$[A]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

   (b) Give an invertible matrix $Q$ such that $Q^{-1}AQ = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

5. Construct solutions to the system
   $$\begin{cases} \dot{x} = ay; \\ \dot{y} = -bx, \end{cases}$$
   where $a$ and $b$ are positive constants, and sketch the phase portrait.