Exam 1 (Part I)

Wednesday, February 18, 2015

Name: ________________________________

This is the in–class portion of Exam 1. This is a closed–book and closed–notes exam. Show all significant work and give reasons for all your answers. Use your own paper and/or the paper provided by the instructor. You have up to 50 minutes to work on the following 2 questions. Relax.

1. Let A denote a 2×2 matrix, and consider the linear system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1}$$

   (a) Suppose that v is an eigenvector of A corresponding to an eigenvalue λ. Verify that the function defined by $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t}\mathrm{v}$, for $t \in \mathbb{R}$, is a solution to the system in (1).

   (b) Suppose that v_1 and v_2 are an eigenvectors of A corresponding to eigenvalues λ_1 and λ_2, respectively. Verify that the function given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t}\mathrm{v}_1 + c_2 e^{\lambda_2 t}\mathrm{v}_2, \quad \text{for } t \in \mathbb{R},$$

   where c_1 and c_2 are arbitrary scalars, is a solution to the system in (1).

   (c) Let v_1, v_2, λ_1 and λ_2 be as in part (b), and assume that v_1 and v_2 are <u>linearly independent</u>. Show that we will always be able to construct a solution of the system in (1) satisfying the initial condition: $x(0) = x_o$, $y(0) = y_o$, for any given values x_o and y_o. Explain your reasoning.

   (d) Sketch the phase portrait of the system in (1) for the specific case in which $\mathrm{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathrm{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\lambda_1 = -1$ and $\lambda_2 = 2$.

2. Consider the first order, linear, non–homogeneous differential equation

$$\frac{dy}{dt} = -2y + e^{-t}, \quad \text{for all } t \in \mathbb{R}. \tag{2}$$

   (a) Construct solutions to the equation in (2).

   (b) Construct a solution to the differential equation in (2) satisfying the initial condition $y(0) = 0$.