Math 102. Rumbos Spring 2015

Solutions to Exam 1 (Part I).

(1) (a) Let v denote an eigenvector of the 2×2 matrix A corresponding to an eigenvalue λ. Define $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} v$$

Then
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{d}{dt}\left[e^{\lambda t} v \right]$$
$$= \lambda e^{\lambda t} v$$
$$= e^{\lambda t}(\lambda v)$$
$$= e^{\lambda t} A v,$$

since $Av = \lambda v$, given that v is an eigenvector of A corresponding to λ. Then

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A\left(e^{\lambda t} v \right)$$
$$= A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

by the definition of $\begin{pmatrix} x \\ y \end{pmatrix}$. Hence $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} v$, for all t, is a solution of

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) Suppose that v_1 & v_2 are eigenvectors of a 2×2 matrix, A, corresponding to eigenvalues λ_1 & λ_2, respectively. Then, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, and v_1 and v_2 are nonzero vectors.

Let
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2, \quad \forall t$$
Then, by the result of part (a)

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{d}{dt}\left[c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \right]$$
$$= c_1 \frac{d}{dt}\left[e^{\lambda_1 t} v_1\right] + c_2 \frac{d}{dt}\left[e^{\lambda_2 t} v_2\right]$$
$$= c_1 A\left(e^{\lambda_1 t} v_1\right) + c_2 A\left(e^{\lambda_2 t} v_2\right)$$
$$= A\left[c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2\right]$$
$$= A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \text{ for all } t,$$

which shows that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ solves $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

(c) Let v_1, v_2, λ_1 and λ_2 be as in part (b) and assume that v_1 & v_2 are linearly independent. By the result of part (b),

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t}, \text{ for } t \in \mathbb{R},$$

solves $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$.

Given any $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}$, we seek c_1 and c_2 such that

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

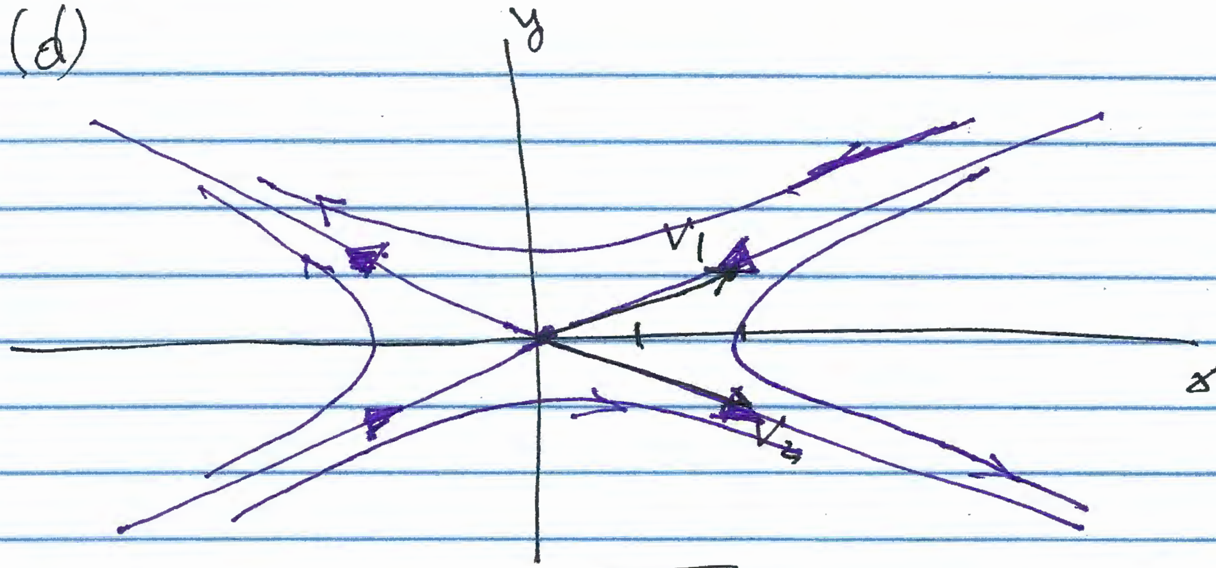
Thus $c_1 v_1 + c_2 v_2 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Now, since v_1 & v_2 are linearly independent, the set $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$. Thus, any $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ is in the span of $\{v_1, v_2\}$; so the there exist c_1 & c_2 such that

$$c_1 v_1 + c_2 v_2 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

which was to be shown.

(d)



2(a) Write equation (2) as

$$\frac{dy}{dt} + 2y = e^{-t}$$

and multiply on both sides by e^{2t} to get

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = e^{t}$$

or
$$\frac{d}{dt}\left[e^{2t}y\right] = e^{t}$$

Integrate to get $e^{2t}y = e^{t} + C$, from which we get that

$$y(t) = e^{-t} + Ce^{-2t}, \quad \text{for all } t \in \mathbb{R}.$$

(b) $y(0) = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$

Then

$$y(t) = e^{-t} - e^{-2t}, \quad \text{for all } t \in \mathbb{R}.$$