Solutions to Review Problems for Exam 2

1. Compute the fundamental matrix for the system

$$\begin{cases} \dot{x} = -3x - y; \\ \dot{y} = 4x - 3y \end{cases} \tag{1}$$

Determine the nature of the stability of the equilibrium point $(0,0)$.

***Solution**:* Write the system in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix} -3 & -1 \\ 4 & -3 \end{pmatrix}. \tag{2}$$

The characteristic polynomial of A in (2) is

$$p_{_A}(\lambda) = \lambda^2 + 6\lambda + 13,$$

which can be written as

$$p_{_A}(\lambda) = (\lambda^2 + 6\lambda + 9) + 4,$$

or

$$p_{_A}(\lambda) = (\lambda + 3)^2 + 4. \tag{3}$$

It follows from (3) that the eigenvalues of A in (2) are

$$\lambda_1 = -3 + 2i \quad \text{and} \quad \lambda_2 = -3 - 2i \tag{4}$$

We look for an invertible matrix Q such that

$$Q^{-1}AQ = J,$$

where

$$J = \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix} \tag{5}$$

In order to do this, we first find an eigenvector $w_1 \in \mathbb{C}^2$ corresponding to λ_1 in (4). We get

$$w_1 = \begin{pmatrix} i \\ 2 \end{pmatrix}.$$

Then, set

$$v_1 = \mathrm{Im}(w_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \mathrm{Re}(w_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \tag{6}$$

so that

$$Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{7}$$

The fundamental matrix, E_J associated with J in (5) is

$$E_J(t) = e^{-3t} \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

Using (5), (6) and (8), we can compute the fundamental matrix corresponding to A by using

$$E_A(t) = QE_J(t)Q^{-1}, \quad \text{for all } t \in \mathbb{R}.$$

We get

$$E_A(t) = e^{-3t} \begin{pmatrix} \cos 2t & -\frac{1}{2}\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

is the fundamental matrix for the system in (1).

Since the eigenvalues of A in (4) are complex with negative real part, $(0,0)$ is a spiral sink. $\square$

2. Compute the general solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{9}$$

and describe the nature of the stability of its equilibrium point. Sketch the phase portrait.

***Solution**:* We first compute the fundamental matrix for the system in (9).

Set

$$A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}. \tag{10}$$

The characteristic polynomial of A in (10) is

$$p_A(\lambda) = \lambda^2 + 6\lambda + 9,$$

which we can write as

$$p_{_A}(\lambda) = (\lambda + 3)^2.$$

Thus,

$$\lambda = -3 \tag{11}$$

is the only eigenvalue of the matrix A in (10).

Next, we find an eigenvector corresponding to $\lambda = -3$, by solving the homogeneous system

$$(A - \lambda I)\mathrm{v} = \mathbf{0}, \tag{12}$$

with $\lambda = -3$. We get the vector

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{13}$$

There is no basis for $\mathbb{R}^2$ made up of eigenvectors of A; therefore, A is not diagonalizable. We therefore need to find a solution, v_2, of the nonhomogeneous system

$$(A - \lambda I)\mathrm{v} = \mathrm{v}_1, \tag{14}$$

with $\lambda = -3$. A solution of (14) is

$$\mathrm{v}_2 = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}. \tag{15}$$

Set $Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix}$, where v_1 and v_2 are given in (13) and (15), respectively; so that,

$$Q = \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

Next, set

$$J = Q^{-1}AQ, \tag{17}$$

where

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{18}$$

It follows from (10), (17), (16) and (18) that

$$J = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}. \tag{19}$$

The fundamental matrix, $E_{_J}(t)$, corresponding to the matrix J in (19) is given by

$$E_{_J}(t) = \begin{pmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}. \tag{20}$$

The fundamental matrix corresponding to the matrix A in (10) is then given by

$$E_{_A}(t) = QE_{_J}(t)Q^{-1}, \quad \text{for all } t \in \mathbb{R},$$

where Q, $E_{_J}(t)$ and Q^{-1} are given in (16), (20) and (18), respectively. We obtain

$$E_{_A}(t) \;=\; \begin{pmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & e^{-3t} - 4te^{-3t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{21}$$

The general solution of the system in (9) is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_{_A}(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

for constants c_1 and c_2, and where $E_{_A}(t)$ is given in (21).

Since the only eigenvalue of A in (11) is negative, it follows that $(0,0)$ is asymptotically stable.

A sketch of the phase portrait of the system in (9) is shown in Figure 1. □

3. Find the equilibrium point of the system

$$\begin{cases} \dot{x} = 2x + y + 1; \\ \dot{y} = x - 2y - 1, \end{cases} \tag{22}$$

and determine the nature of the stability of the point. Sketch the phase portrait.

***Solution**:* We first determine the nullclines:

$$\begin{array}{ll} \dot{x} = 0\text{–nullcline:} & 2x + y = -1 \\ \dot{y} = 0\text{–nullcline:} & x - 2y = 1 \end{array}$$

These lines are sketched in Figure 2. The nullclines intersect at the equilibrium point

$$(\overline{x}, \overline{y}) = \left(-\frac{1}{5}, -\frac{3}{5}\right). \tag{23}$$

The derivative of the vector field

$$F(x,y) = \begin{pmatrix} 2x + y + 1 \\ x - 2y - 1 \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2,$$

at the equilibrium point $(\overline{x}, \overline{y})$ in (23) is

$$DF(\overline{x}, \overline{y}) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

which has eigenvalues

$$\lambda = \pm\sqrt{5}.$$

Hence, $(\overline{x}, \overline{y})$ is a saddle point for the system in (22). A sketch of the phase portrait of the system in (22) is shown in Figure 2. $\square$

4. Let A denote a 2×2 matrix satisfying $\det A < 0$.

   (a) Explain why the origin is an isolated equilibrium point of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{24}$$

***Solution**:* The equilibrium points of the system (24) are solutions of the algebraic system

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{25}$$

Since we are assuming that $\det(A) < 0$, the matrix A is invertible. Therefore, the system in (25) has

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as its only solution. Therefore, the origin is an isolated equilibrium point of the system in (24) $\square$

   (b) Determine the nature of the stability or unstability of the origin for the system in (24). Explain your reasoning.

***Solution**:* The characteristic polynomial of the matrix A is

$$p_{_A}(\lambda) = \lambda^2 - \text{trace}(A)\ \lambda + \det(A).$$

Therefore, the eigenvalues of A are given by

$$\lambda = \frac{\text{trace}(A) \pm \sqrt{\text{trace}(A)^2 - 4\det(A)}}{2}. \tag{26}$$

It follows from (26) and the assumption that $\det(A) < 0$ that the eigenvalues of A are real and distinct; call the eigenvalues λ_1 and λ_2. Furthermore, since $\det(A) = \lambda_1\lambda_2$, it also follows from the assumption that $\det(A) < 0$ that λ_1 and λ_1 have opposite signs. Hence, the origin is a saddle point for the system in (**??**24). $\square$

5. Find two distinct solutions of the initial value problem

$$\begin{cases} \dot{x} = 6tx^{2/3}; \\ x(0) = 0. \end{cases} \tag{27}$$

Why doesn't this violate the uniqueness assertion of the local existence and uniqueness theorem?

Solution: Use separation of variables to show that the function

$$x_1(t) = t^6, \quad \text{for all } t \in \mathbb{R},$$

solves the initial value problem (IVP) in (27).

Verify that the function

$$x_2(t) = 0, \quad \text{for all } t \in \mathbb{R},$$

also solves the IVP in (27).

Thus, the IVP in (27) has at least two distinct solutions.

Observe that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,t) = 6tx^{2/3}, \quad \text{for } (x,t) \in \mathbb{R}^2,$$

does not have a continuous partial derivative with respect to x at $(0,0)$. Indeed, for $t \neq 0$ and $x \neq 0$,

$$\frac{\partial f}{\partial x} = \frac{4t}{x^{1/3}}$$

does not have a limit as (x,t) approaches $(0,0)$. Hence, the local existence and uniqueness theorem discussed in class does not apply to the IVP (27). □

6. Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y^2 - y; \\ \\ y(0) = 2. \end{cases} \tag{28}$$

Give the maximal interval of existence for the solution. Does the solution exist for all t? If not, explain what prevents the solution from being extended further.

Solution: Use separation of variables and partial fractions to derive the solution

$$y(t) = \frac{2}{2 - e^t}, \quad \text{for } t < \ln(2). \tag{29}$$

Note that the denominator of the expression in (29) is 0 when $t = \ln(2)$. At that time the solution to the IVP in (28) given in (29) ceases to exist. Hence, the maximal interval of existence for the solution of the IVP in (28) is $(-\infty, \ln(2))$.
$\square$

7. The motion of an object of mass m, attached to a spring of stiffness constant k, and moving along a horizontal flat surface is modeled by the second–order, linear differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0, \tag{30}$$

where $x = x(t)$ denotes the position of the object along its direction of motion, and γ is the coefficient of friction between the object and the surface.

(a) Express the equation in (30) as a system of first order linear differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{31}$$

***Solution**:* The matrix A in (31) is given by

$$A = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix} \tag{32}$$

$\square$

(b) For the matrix A in (31), let $\omega^2 = \dfrac{k}{m}$ and $b = \dfrac{\gamma}{2m}$.
Give the characteristic polynomial of the matrix A, and determine when the A has (i) two real and distinct eigenvalues; (ii) only one real eigenvalue; (iii) complex eigenvalues with nonzero imaginary part.
***Solution**:* The matrix A in (32) can now be written as

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2b \end{pmatrix} \tag{33}$$

The characteristic polynomial of the matrix A in (33) is then

$$p_{_A}(\lambda) = \lambda^2 + 2b\lambda + \omega^2.$$

Thus, the eigenvalues of the matrix A in (33) are given by

$$\lambda = -b \pm \sqrt{b^2 - \omega^2}.$$

Thus, A has

(i) two real and distinct eigenvalues, if $b > \omega$;

(ii) only one real eigenvalue, if $b = \omega$;

(iii) complex eigenvalues with nonzero imaginary part, if $b < \omega$.

$\square$

(c) Describe the behavior of solutions of (30) in case (iii) of part (b).

Solution: If $b < \omega$, the eigenvalues of A are complex with negative real part. Hence, the solutions of the equation (30) will oscillate with decreasing amplitude. $\square$

8. The following system of first order differential equations can be interpreted as describing the interaction of two species with population densities x and y:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - y); \\ \\ \dfrac{dy}{dt} = y(0.5 - 0.25y - 0.75x) \end{cases} \tag{34}$$

(a) What do these equations predict about the population density of each species if the other were not present? What effect do the species have on each other? Describe the kind of interaction that this system models.

Solution: In the absence of the species of density y, the species of density x will grow according to the differential equations

$$\frac{dx}{dt} = x(1 - x),$$

which is a logistic equation with intrinsic growth rate $r = 1$ and carrying capacity $L = 1$. Thus, in the absence of the species of density y, the species of density x will experience logistic growth.

Similarly, in the absence of the species of density x, the species of density y will grow according to the equation

$$\frac{dy}{dt} = y(0.5 - 0.25y),$$

or

$$\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{2}\right),$$

which is a logistic equation with intrinsic growth rate $r = 0.5$ and carrying capacity $L = 2$. Hence, in the absence of the species of density x, the species of density y will also experience logistic growth.

When both species are present, they each have a negative effect on the *per capita* growth rate of the other in the sense that

$$\frac{1}{x}\frac{dx}{dt} = 1 - x - y$$

and

$$\frac{1}{y}\frac{dy}{dt} = 0.5 - 0.25y - 0.75x;$$

so that both rates get reduced because of the presence of the other species. Thus, the two species are in competition with each other. Hence, the system of differential equations in (34) models competition between the species. □

(b) Sketch the nullclines, determine the equilibrium points, sketch some possible trajectories, and determine the nature of the stability of all the equilibrium points.

***Solution**:* The $\dot{x} = 0$–nulclines are the lines

$$x = 0 \ \ \text{(the } y\text{–axis)} \quad \text{and} \quad x + y = 1;$$

and the $\dot{y} = 0$–nullclines are the lines

$$y = 0 \ \ \text{(the } x\text{–axis)} \quad \text{and} \quad 3x + y = 2.$$

These are sketched in Figure 3.

Distinct types of nullclines meet at four points in the first quadrant

$$(0,0), \quad (1,0), \quad (0,2), \quad \text{and} \quad \left(\frac{1}{2}, \frac{1}{2}\right). \tag{35}$$

These are the equilibrium points.

We next apply the Principle of Linearized Stability at each of the equilibrium points in (35). In order to do this, we first compute the derivative of the vector field

$$F(x,y) = \begin{pmatrix} x - x^2 - xy \\ 0.5y - 0.25y^2 - 0.75xy \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2;$$

namely,

$$DF(x,y) = \begin{pmatrix} 1 - 2x - y & -x \\ -0.75y & 0.5 - 0.5y - 0.75x \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{36}$$

Evaluating the derivative map in (36) at the equilibrium point (0.0) yields

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix},$$

which has positive eigenvalues, 1 and 0.5. Thus, $(0,0)$ is a source; so that, $(0,0)$ is unstable.

At the equilibrium point $(1,0)$ we obtain

$$DF(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix},$$

which has negative eigenvalues, -1 and -0.25; so that $(1,0)$ is a sink, and therefore asymptotically stable.

At the equilibrium point $(0,2)$ we obtain

$$DF(0,2) = \begin{pmatrix} -1 & 0 \\ -1.5 & -0.5 \end{pmatrix},$$

which has negative eigenvalues, -1 and -0.5; so that $(0,2)$ is also a sink, and therefore asymptotically stable.

Evaluating the derivative map in (36) at the fourth equilibrium point in (35) we obtain

$$DF(1/2,1/2) = \begin{pmatrix} -1/2 & -1/2 \\ -3/8 & -1/8 \end{pmatrix},$$

which has determinant $-\dfrac{1}{8} < 0$; so that, by the result of Problem 4, the origin is a saddle point of the linearization of the system in (34) at the equilibrium point $(1/2,1/2)$. Hence, by the Principle of Linearized Stability, $(1/2,1/2)$ is a saddle point for the system in (34).

A sketch of the phase portrait of the system in (34) picturing a few trajectories is shown in Figure 3. $\square$

(c) Describe the different possible long–run behaviors of x and y as $t \to \infty$, and interpret the result in terms of the populations of the two species.

***Solution**:* An inspection of the sketch of the phase portrait of the system in (34) shown in Figure 3 reveals that two things can happen. Depending on the initial conditions, the trajectory might end up up the sink $(1,0)$, indicating that the species of density x will survive the competition, while the species of density y will go extinct. On the other hand, a trajectory might might end up at the sink $(0,2)$; in which case the species of density

x will go extinct, while the species of density y will survive. Since the the equilibrium point $(0.5, 0.5)$ is unstable (a saddle point), there will not be coexistence of the species; one will outcompete the other to extinction. $\square$

9. Let Ω denote an open interval of real numbers, and $f\colon \Omega \to \mathbb{R}$ denote a continuous function. Let $x_p\colon \Omega \to \mathbb{R}$ denote a particular solution of the nonhomogeneous, second–order equation
$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \quad \text{for } t \in \Omega, \tag{37}$$
where b and c are real constants.
   (a) Let $x\colon \Omega \to \mathbb{R}$ denote any solution of (37) and put
   $$u(t) = x(t) - x_p(t), \quad \text{for } t \in \Omega.$$
   Verify that u solves the homogeneous, second–order equation
   $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0, \quad \text{for } t \in \Omega, \tag{38}$$
   (b) Let $x_1\colon \Omega \to \mathbb{R}$ and $x_2\colon \Omega \to \mathbb{R}$ denote linearly independent solutions of the homogenous equation (38). Prove that any solution of the nonhomogeneous equation in (37) must be of the form
   $$x(t) = c_1x_1(t) + c_2x_2(t) + x_p(t), \quad \text{for all } t \in \Omega,$$
   where c_1 and c_2 are constants.

10. Consider the two–dimensional system
$$\begin{cases} \dot{x} = y + x(x^2 + y^2) \\ \dot{y} = -x + y(x^2 + y^2) \end{cases} \tag{39}$$
   (a) Show that $(0, 0)$ is an isolated critical point of the system in (39).
   (b) Compute solutions to the linearization of the system in (39) around the origin.
   (c) Determine the nature of the stability of the origin for the linearized system.
   (d) Let $r^2 = x^2 + y^2$ and note that $r\dfrac{dr}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$.
   Show that the solutions to the system in (39) with initial condition $r(0) = r_o > 0$ becomes unbounded as $t \to 1/2r_o^2$, and hence the equilibrium point $(0, 0)$ for the system in (39) is unstable.
   (e) Explain why (c) and (d) together do not contradict the Principle of Linearized Stability.

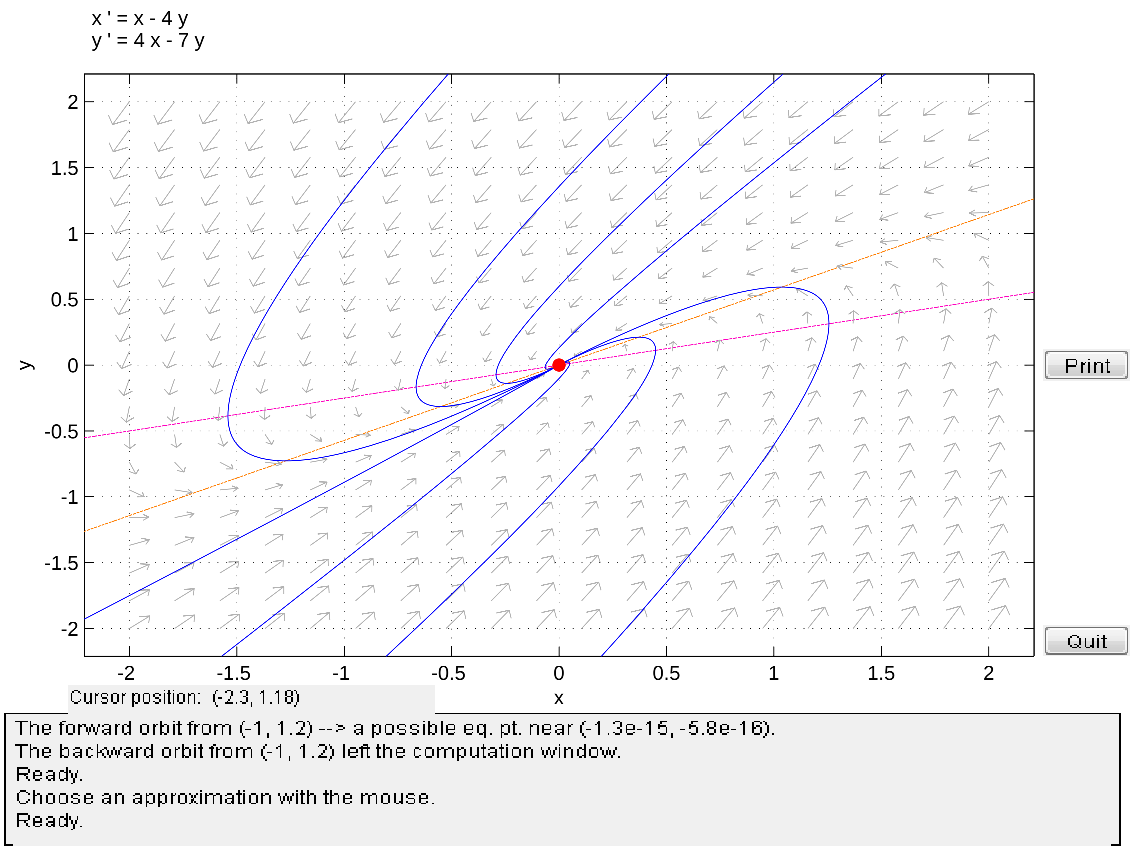

Figure 1: Sketch of Phase Portrait for System (9)

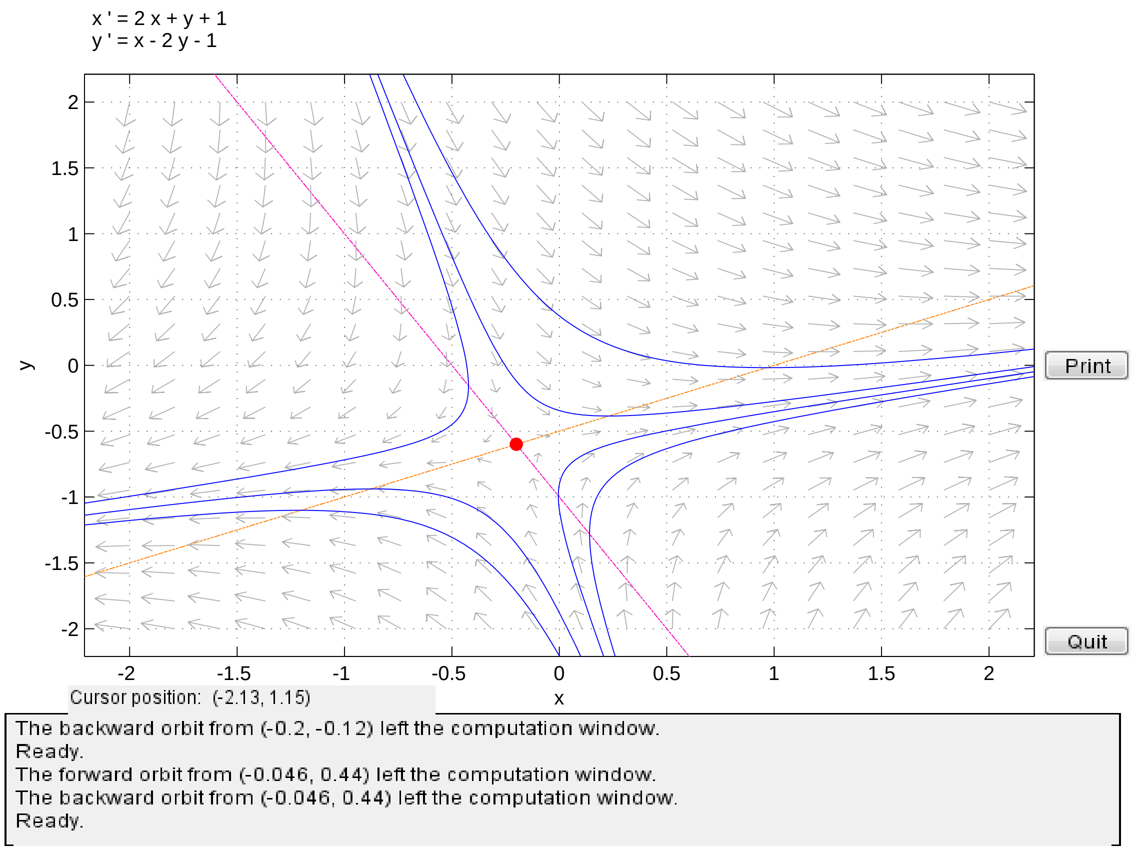

Figure 2: Sketch of Phase Portrait for System (22)

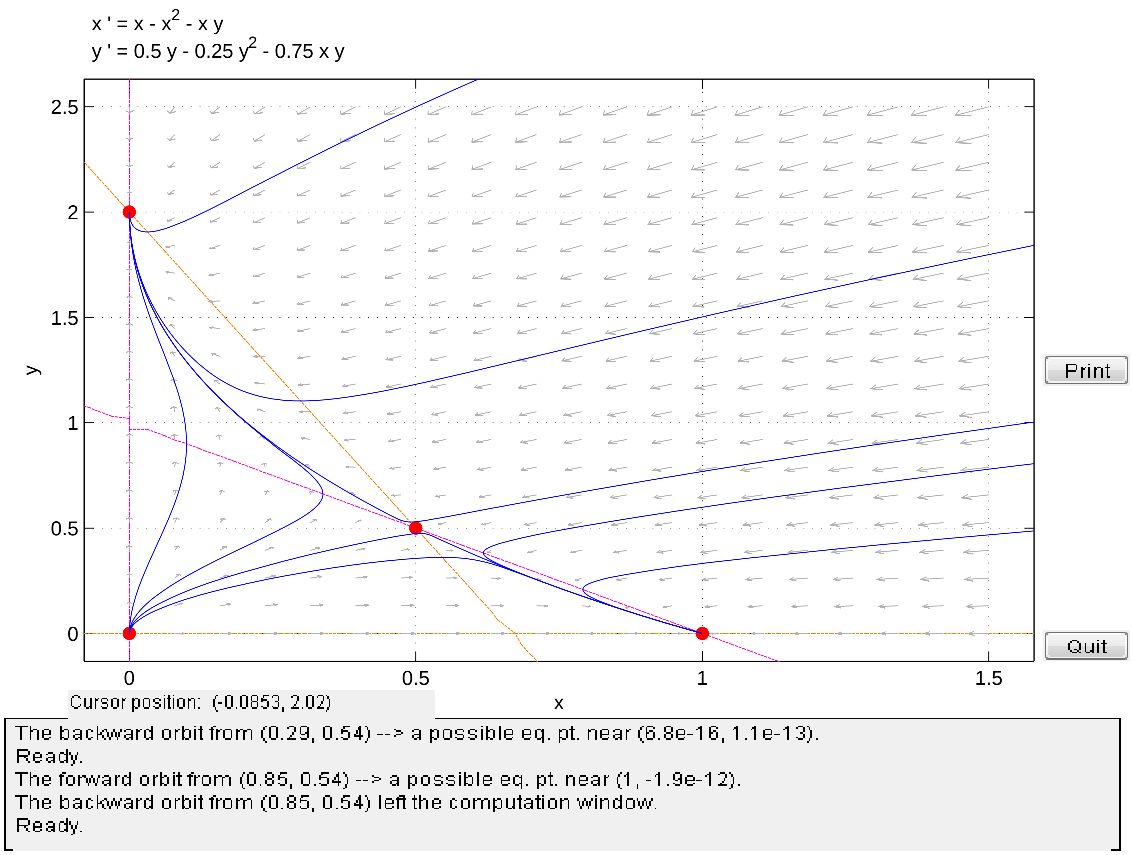

Figure 3: Sketch of Phase Portrait of System in (34)