Review Problems for Final Exam

1. Consider a chemical reaction
$$A + B \rightarrow C$$
in which two substances, A and B, react to produce a single substance, C. Assume that the reverse reaction does not have a considerable effect and therefore can be neglected. Let $y = y(t)$ denote the number of kilograms of the reaction product, C, after t minutes. Suppose that the original amount of the reacting substances are 80 kilograms and 60 kilograms. As a consequence of the law of mass action, we obtain that
$$\frac{dy}{dt} = k(80 - y)(60 - y) \quad \text{for some constant } k > 0.$$
That is, the rate of production of C is proportional to the product of the remaining amounts of the reactants A and B.

   (a) Sketch some possible solutions to the equation.

   (b) Use separation of variables to solve the above differential equation assuming that $y = 0$ when $t = 0$.

   (c) In part (b), assume also that there are 20 kilograms of the reaction product 10 minutes after the onset of the reaction. How much reaction product is present 5 minutes later?

2. The differential equation
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - p(N, \Lambda), \tag{1}$$
models a population that is subject to predation reflected in the term $p(N, \Lambda)$, which depends on the population size, N, and a set of parameters, Λ. In the absence of predation the population undergoes logistic growth with intrinsic growth rate, r, and carrying capacity, K.

   In 1978, Ludwig, Jones and Holling published an article in the Journal of Animal Ecology (*Qualitative analysis of insect outbreak systems: the spruce budworm and forest*, Volume 47, pp. 315–332) in which they proposed the following constitutive equation for the predation term,
$$p(N, a, b) = \frac{bN^2}{a^2 + N^2}. \tag{2}$$

(a) Give interpretations for the parameters a and b in (2).

(b) Nondimensionalize the differential equation in (1) by introducing dimensionless variables

$$u = \frac{N}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda},$$

to obtain the dimensionless equation

$$\frac{du}{d\tau} = \alpha u \left(1 - \frac{u}{\beta}\right) - \frac{u^2}{1+u^2}, \tag{3}$$

where α and β are dimensionless parameters.

Express α and β in terms of the parameters r, K, a and b.

(c) Observe that $u = 0$ is an equilibrium point of the equation in (3). Determine the nature of the stability of this equilibrium point.

3. In this problem we show how small changes in the coefficients of system of linear equations can affect stability of an equilibrium point that is a center.

   (a) Consider the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Show that $(0, 0)$ a center.

   (b) Next, consider $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, where $|\varepsilon| \neq 0$ is arbitrarily small. Show that no matter how small $|\varepsilon| \neq 0$ is, the center in part (a) becomes a spiral point. Discuss the stability–type for $\varepsilon > 0$ and for $\varepsilon < 0$.

4. Consider the second order, linear, homogeneous differential equation

$$\frac{d^2x}{dt^2} + \mu x = 0, \tag{4}$$

   where μ is a real parameter.

   (a) Give the general solution for each of the cases (i) $\mu < 0$, (ii) $\mu = 0$ and (iii) $\mu > 0$.

   (b) For each of the cases (i), (ii) and (iii) in part (a), determine conditions on μ (in any) that will guarantee that the equation in (4) has a nontrivial solution $x \colon \mathbb{R} \to \mathbb{R}$ satisfying $x(0) = 0$ and $x(\pi) = 0$.

5. Give the general solution of the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

6. Consider the the nonlinear differential equation
$$\frac{du}{dt} = e^u - 1.$$
Find the equilibrium points of the equations and study their stability.

7. Consider the two–dimensional, autonomous system
$$\begin{cases} \dfrac{dx}{dt} & = & (x-y)(1-x^2-y^2); \\ \\ \dfrac{dy}{dt} & = & (x+y)(1-x^2-y^2). \end{cases}$$
   (a) Verify that every point in the unit circle, $C = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$, is an equilibrium point.
   (b) Show that $(0,0)$ is an isolated equilibrium point of the system.
   (c) Determine the nature of the stability of $(0,0)$.
   (d) Let D denote the open unit disc in $\mathbb{R}^2$,
   $$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}.$$
   Show that every trajectory that starts at a point $(x_o, y_o) \in D$, such that $(x_o, y_o) \neq (0,0)$, will tend towards C as $t \to \infty$.
   (e) Show that every trajectory that starts at a point $(x_o, y_o) \in \mathbb{R}^2$, such that $x_o^2 + y_o^2 > 1$, will tend towards C as $t \to \infty$.

8. Consider the two–dimensional, autonomous system
$$\begin{cases} \dfrac{dx}{dt} & = & x - y - x(x^2+y^2); \\ \\ \dfrac{dy}{dt} & = & x + y - y(x^2+y^2). \end{cases}$$
   (a) Show that $(0,0)$ is an isolated equilibrium point of the system.
   (b) Determine the nature of the stability of $(0,0)$.

9. Consider the two–dimensional, autonomous system
$$\begin{cases} \dfrac{dx}{dt} & = & y; \\ \\ \dfrac{dy}{dt} & = & 4x - x^3. \end{cases}$$

(a) Sketch nullclines, compute equilibrium points, and use the Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

(b) Find a conserved quantity for the system.

(c) Discuss the phase–portrait of the system.

10. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= x(2 - x - y); \\ \\ \dfrac{dy}{dt} &= y(3 - 2x - y) \end{cases}$$

describes competing species of densities $x \geqslant 0$ and $y \geqslant 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

11. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= \dfrac{c}{a + ky} - b; \\ \\ \dfrac{dy}{dt} &= \gamma x - \beta, \end{cases}$$

models the time evolution of the interaction of an enzyme of concentration, y, and m–RNA, of concentration x, in a process of protein synthesis. The parameters a, b, c, k, α and β are assumed to be positive. This model was proposed by Brian C. Goodwin in 1965 (*Oscillatory in Enzymatic Control Processes,* in Advances in Enzyme Regulation,Volume 3, 1965, Pages 425–428, IN1–IN2, 429430, IN3–IN6, 431–437).

(a) Sketch the nullclines, find all equilibrium points, and apply the Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

(b) Find a conserved quantity for the the system.

(c) Discuss the phase–portrait of the system.