Assignment #19

Due on Friday, April 24, 2015

Read on *The Principle of Linearized Stability,* in the class lecture notes at `http://pages.pomona.edu/`

Background and Definitions.

The Principle of Linearized Stability. For a the system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\ \\ \dfrac{dy}{dt} = g(x,y), \end{cases} \tag{1}$$

here $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are continuous functions defined on an open subset, D, of $\mathbb{R}^2$, and which have continuous partial derivatives in D, an equilibrium point, $(\overline{x},\overline{y})$, is a solution of the system of equations

$$\begin{cases} f(x,y) = 0; \\ g(x,y) = 0. \end{cases}$$

Set

$$F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \quad \text{for all} \quad (x,y) \in D,$$

The linearization of the system in (1) around an equilibrium point $(\overline{x},\overline{y})$ is the linear system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = DF(\overline{x},\overline{y}) \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2}$$

where

$$DF(\overline{x},\overline{y}) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(\overline{x},\overline{y}) & \dfrac{\partial f}{\partial y}(\overline{x},\overline{y}) \\ \\ \dfrac{\partial g}{\partial x}(\overline{x},\overline{y}) & \dfrac{\partial g}{\partial y}(\overline{x},\overline{y}) \end{pmatrix}.$$

The **Principle of Linearized Stability** states that, for the case in which

$$\det[DF(\overline{x},\overline{y})] \neq 0,$$

and the eigenvalues of the matrix $DF(\overline{x},\overline{y})$ have nonzero real part, then phase portrait of the system in (1) near an equilibrium $(\overline{x},\overline{y})$ looks like the phase portrait of the linear system in (2) near the origin.

Do the following problems

In problems (1)–(5), given the two–dimensional system, (a) sketch the nullclines; (b) determine the critical points; (c) find the derivative of the vector field associated with the system; (d) determine the stability of the origin for each linearized system; (e) use the principle of linearized stability (when applicable) to determine the stability type of each equilibrium point of the non–linear system; and (f) sketch the phase portrait.

1. $$\begin{cases} \dot{x} = -3x + 2xy; \\ \dot{y} = -4y + 3xy. \end{cases}$$

2. $$\begin{cases} \dot{x} = x(1 - 2y); \\ \dot{y} = y(x - 1). \end{cases}$$

3. $$\begin{cases} \dot{x} = y; \\ \dot{y} = x - y - x^3. \end{cases}$$

4. $$\begin{cases} \dot{x} = y - x^3; \\ \dot{y} = y - 4x. \end{cases}$$

5. $$\begin{cases} \dot{x} = x(1 - 2x) - 3y; \\ \dot{y} = y(x - 1). \end{cases}$$