Assignment #5

Due on Monday, February 9, 2015

Read Section 3.3, on *Length along Curves,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 17.2, on *Motion, Velocity and Acceleration,* in Calculus: Multivariable, by McCallum, Hughes–Hallett, Gleason, et al.

Background and Definitions.

- **Norm of Vectors**. Given a vector $\overrightarrow{v} = (a, b)$ in the plane, its **norm** (denoted by $\|\overrightarrow{v}\|$) is defined by
$$\|\overrightarrow{v}\| = \sqrt{a^2 + b^2}.$$
Geometrically, $\|\overrightarrow{v}\|$ gives the length of the vector $\overrightarrow{v}$, or the distance from the tip of $\overrightarrow{v}$ when its tail is placed at the origin $O(0, 0)$.
If $\|\overrightarrow{v}\| = 1$, we say that $\overrightarrow{v}$ is a **unit vector**.

- **Distance between Points**. Given points P and Q in the plane, the distance from P to Q, denoted by $\text{dist}(P, Q)$ is given by
$$\text{dist}(P, Q) = \|\overrightarrow{OQ} - \overrightarrow{OP}\|.$$

- **Arclength**. Let C denote a curve parametrized by the differentiable path $\overrightarrow{r} \colon [a, b] \to \mathbb{R}^2$, where a and b are real numbers with $a < b$. The arclength along the curve C, denoted by $\ell(C)$, is given by
$$\ell(C) = \int_a^b \|\overrightarrow{r}'(t)\| \; dt,$$
provided that the integral exists.

Do the following problems

1. Let J denote an open interval in $\mathbb{R}$, and $\overrightarrow{r} \colon J \to \mathbb{R}^2$ be a differentiable path with continuous derivative $\overrightarrow{r}'$. For fixed $a \in J$, define
$$s(t) = \int_a^t \|\overrightarrow{r}'(\tau)\| \; \mathrm{d}\tau \quad \text{for all} \;\; t \in J.$$
Show that s is differentiable and compute $s'(t)$ for all $t \in J$.

2. Let $\overrightarrow{r}$ and s be as defined in the previous problem. Suppose, in addition, that $\overrightarrow{r}'(t)$ is never the zero vector for all t in J. Show that s is a strictly increasing function of t and that it is, therefore, one–to–one.

3. Let $\overrightarrow{r}$ and s be as defined in Problem 1. We can re–parameterize $\overrightarrow{r}$ by using s as a parameter. We therefore obtain $\overrightarrow{r}(s)$, where s is the *arc length* parameter.

   Differentiate the expression
   $$\overrightarrow{r}(s(t)) = \overrightarrow{r}(t)$$
   with respect to t using the Chain Rule. Conclude that, if $\overrightarrow{r}'(t)$ is never the zero vector for all t in J, then $\overrightarrow{r}'(s)$ is always a unit vector.

   The vector $\overrightarrow{r}'(s)$ is called the *unit tangent vector* to the path $\overrightarrow{r}$.

4. For a and b, positive real numbers, the expression
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
   defines an ellipse in the xy–plane $\mathbb{R}^2$.

   Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length. Do not evaluate the integral.

5. Let $\overrightarrow{r}\colon [-1, 2] \to \mathbb{R}^2$ be defined by $\overrightarrow{r}(t) = t\,\widehat{i} + t^2\,\widehat{j}$ for all $t \in [-1, 2]$. Describe the curve parametrized by $\overrightarrow{r}$ and compute the arc length of the curve.