Review Problems for Exam 3

1. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 1. In the first

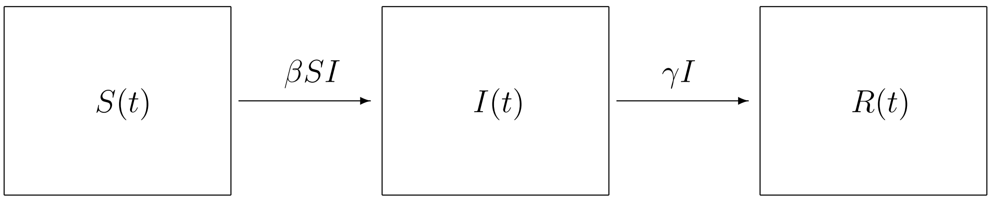

Figure 1: SIR Compartments

compartment, $S(t)$ denotes the number of individuals in a population that are susceptible to acquiring the disease; in the second compartment, $I(t)$ denotes the number of infected individual who can also infect others; and, in the third compartment, $R(t)$ denotes the number of individuals who had the disease and who have recovered from it; they can no longer get infected.

The arrows between compartments indicate the rates at which individuals flow from one compartment to the other. For instance, the arrow between the first two compartments indicates the transmission rate of the disease; it is assumed that the rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is indicated by the arrow between the last two compartments; it is assumed that this rate is proportional to the number of infected individuals, with constant of proportionality $\gamma > 0$.

(a) Use conservation principles to derive a system of differential equations for the functions S, I and R, assuming that they are differentiable, of the form

$$\begin{cases} \dfrac{dS}{dt} = f(S, I, R, \beta, \gamma); \\[2ex] \dfrac{dI}{dt} = g(S, I, R, \beta, \gamma); \\[2ex] \dfrac{dR}{dt} = h(S, I, R, \beta, \gamma), \end{cases} \tag{1}$$

where f, g and h are continuous functions that have continuous partial derivatives with respect to S, I and R. The system in (1) is known in the literature as the Kermack–McKendrick SIR model. It first appeared in the scientific literature in 1927.

(b) Deduce that the system in (1) implies that the total number of individuals in the population,

$$N(t) = S(t) + I(t) + R(t),$$

remains constant. Denote $N(t)$ by N, where N is a constant, for all t.

(c) Explain why the result of part (c) implies that the study of the system (1) reduces to the study of the two–dimensional system

$$\begin{cases} \dfrac{dS}{dt} = f(S, I, R, \beta, \gamma); \\[2ex] \dfrac{dI}{dt} = g(S, I, R, \beta, \gamma). \end{cases} \tag{2}$$

(d) Analyze the system obtained in part (c). What does the model in (1) predict about the spread of the disease in terms of the initial conditions $S(0) = S_o$, $I(0) = I_o$, $R(0) = 0$, and the parameters β, γ and N? Under which conditions will the number of infected individuals increase (an epidemic outbreak), or decrease?

2. For the following systems, sketch nullclines, find equilibrium points, determine their stability properties, and describe the local behavior trajectories near the equilibrium points. Sketch the phase portraits.

(a) $\begin{cases} \dot{x} = x^2 - y^2 - 1; \\ \dot{y} = 2y, \end{cases}$

(b) $\begin{cases} \dot{x} = y - y^2 + 2; \\ \dot{y} = 2x^2 - 2xy, \end{cases}$

(c) $\begin{cases} \dot{x} = 4 - 2y; \\ \dot{y} = 12 - 3x^2. \end{cases}$

3. Sketch the nullclines and find the equilibrium point of the system

$$\begin{cases} \dot{x} = x + y - 1; \\ \dot{y} = -x + y. \end{cases}$$

   Determine the nature of the stability of the equilibrium point. Sketch the phase portrait.

4. Sketch the flow of the linear vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6x + 4y \\ -10x - 6y \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

   Suggestion: Sketch nullclines and determine the nature of the stability of the origin.

5. The following system of first order differential equations can be interpreted as describing the interaction of two species with population densities x and y:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - 2x) - 0.5xy; \\ \\ \dfrac{dy}{dt} = 0.5xy - 0.5y. \end{cases}$$

   (a) What do these equations predict about the population density of each species if the other were not present? What effect do the species have on each other? Describe the kind of interaction that this system models.

   (b) Sketch the nullclines, determine the equilibrium points in the first quadrant, apply the principle of linearized stability (when applicable) to determine the nature of the stability of all the equilibrium points found in part (b), and sketch some possible trajectories.

   (c) Describe the different possible long–run behaviors of x and y as $t \to \infty$, and interpret the result in terms of the populations of the two species.