Review Problems for Final Exam

1. Let $f(x,y) = x^2 - y^2$ for all $(x,y) \in \mathbb{R}^2$.

   (a) Compute the gradient field $F(x,y) = \nabla f(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

   (b) Sketch the flow of the vector field $F(x,y)$ given in part (a).

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a real valued function with continuous second partial derivatives. Define the negative gradient vector field

$$F(x,y) = -\nabla f(x,y), \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{1}$$

   (a) Let $(x(t), y(t))$ denote a flow curve of the Field given in (1) that contains no equilibrium points of (1) the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla f(x,y). \tag{2}$$

   Show that f is strictly decreasing (with increasing t) along this trajectory.

   (b) Let $(x(t), y(t))$ denote a solution curve of the system in (2) that contains no equilibrium points of (2). Explain why this trajectory cannot be a cycle (a closed curve, or a loop).

3. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= x(2 - x - y); \\ \\ \dfrac{dy}{dt} &= y(3 - 2x - y) \end{cases}$$

   describes competing species of densities $x \geqslant 0$ and $y \geqslant 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

4. Let C denote the ellipse given by the equation

$$4x^2 + y^2 = 4,$$

   and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the linear function given by

$$f(x,y) = 4x + 7y, \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

   Find points on C at which the gradient of f is perpendicular to C.

   Suggestion: Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = 4x^2 + y^2$, for all $(x,y) \in \mathbb{R}^2$. Observe that C is a level set of g.

5. Consider the Lotka–Volterra system

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy; \\ \\ \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{3}$$

where the parameters α, β, γ and δ are assumed to be positive constants. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, and define $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(x, y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y), \quad \text{for } (x, y) \in D. \tag{4}$$

(a) Compute the partial derivatives

$$\frac{\partial H}{\partial x}, \quad \frac{\partial H}{\partial y}, \quad \frac{\partial^2 H}{\partial x^2}, \quad \frac{\partial^2 H}{\partial y \partial x}, \quad \frac{\partial^2 H}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 H}{\partial y^2},$$

for $(x, y) \in D$.

(b) Find points in D at which the gradient of H is the zero vector.

(c) Let $(x(t), y(t)$ denote a solution curve of the Lotka–Volterra system in (3). Show that the function H defined in (4) is constant on the curve.

Suggestion: Use the Chain Rule to compute

$$\frac{d}{dt}[H(x(t), y(t))].$$

(d) Verify that the system in (3) has only one equilibrium point in D; call it $(\overline{x}, \overline{y})$.

(e) Show that H has a minimum value at the equilibrium point $(\overline{x}, \overline{y})$ found in part (d). Conclude therefore that the solution curves of the system in (3) near $(\overline{x}, \overline{y})$ are closed curves. Hence $(\overline{x}, \overline{y})$ is a center.