Math 32S, Rumbos Spring 2015

Solutions to Review Problems for Final Exam

1. $f(x,y) = x^2 - y^2$

(a) $F(x,y) = \nabla f(x,y) = 2x\,\hat{\imath} - 2y\,\hat{\jmath}$

or

$$F(x,y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$$

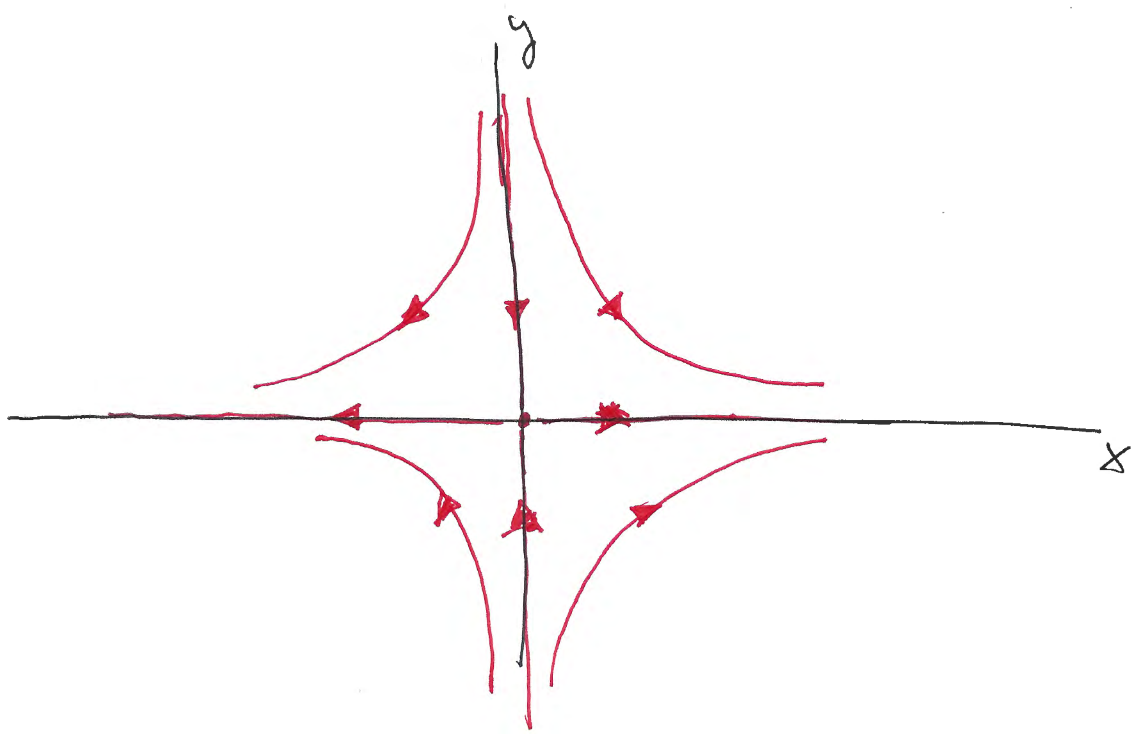
(b) The flow of F are solutions of the system

$$\begin{cases} \dot{x} = 2x; \\ \dot{y} = -2y, \end{cases}$$

which has solution curves

$$(x(t), y(t)) = (c_1 e^{2t}, c_2 e^{-2t}) \quad t \in \mathbb{R}.$$

A few of these are sketched below.



2. $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = -\nabla f(x,y), \quad (x,y) \in \mathbb{R}^2.$$

(a) Let $(x(t), y(t))$ be a flow curve for F, so that

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla f(x,y), \qquad (2.1)$$

or

$$\frac{dx}{dt} = -\frac{\partial f}{\partial x}(x(t), y(t)), \quad \text{for all } t$$

$$\frac{dy}{dt} = -\frac{\partial f}{\partial y}(x(t), y(t)), \quad \text{for all } t.$$

Consider $f(x(t), y(t))$ for all t.
Use the Chain Rule to compute

$$\frac{d}{dt}[f(x(t), y(t))] = \nabla f(x(t), y(t)) \cdot (x'(t), y'(t))$$

$$= \nabla f(x(t), y(t)) \cdot (-\nabla f(x(t), y(t)))$$

$$= -\|\nabla f(x(t), y(t))\|^2$$

Thus, $\frac{d}{dt}[f(x(t), y(t))] \leq 0$ for all t.

Since $(x(t), y(t))$ has no equilibrium points of the system (2.1), it follows that $\nabla f(x(t), y(t)) \neq 0$ for all t. Thus $\frac{d}{dt}[f(x(t), y(t))] = -\|\nabla f(x(t), y(t))\|^2 < 0$; so that f is strictly decreasing on this curve.

2.(b) If $(x(t), y(t))$ were a cycle, then there would be $T > 0$ such that

$$(x(0), y(0)) = (x(T), y(T)). \qquad (2.2)$$

Thus, since $f(x(t), y(t))$ is strictly decreasing on $(x(t), y(t))$, by part (a),

$$f(x(T), y(T)) < f(x(0), y(0)).$$

This is impossible by virtue of (2.2). Hence, $(x(t), y(t))$ cannot be a cycle. $\blacksquare$

3. $\begin{cases} \dot{x} = x(2-x-y) \\ \dot{y} = y(3-2x-y) \end{cases}$

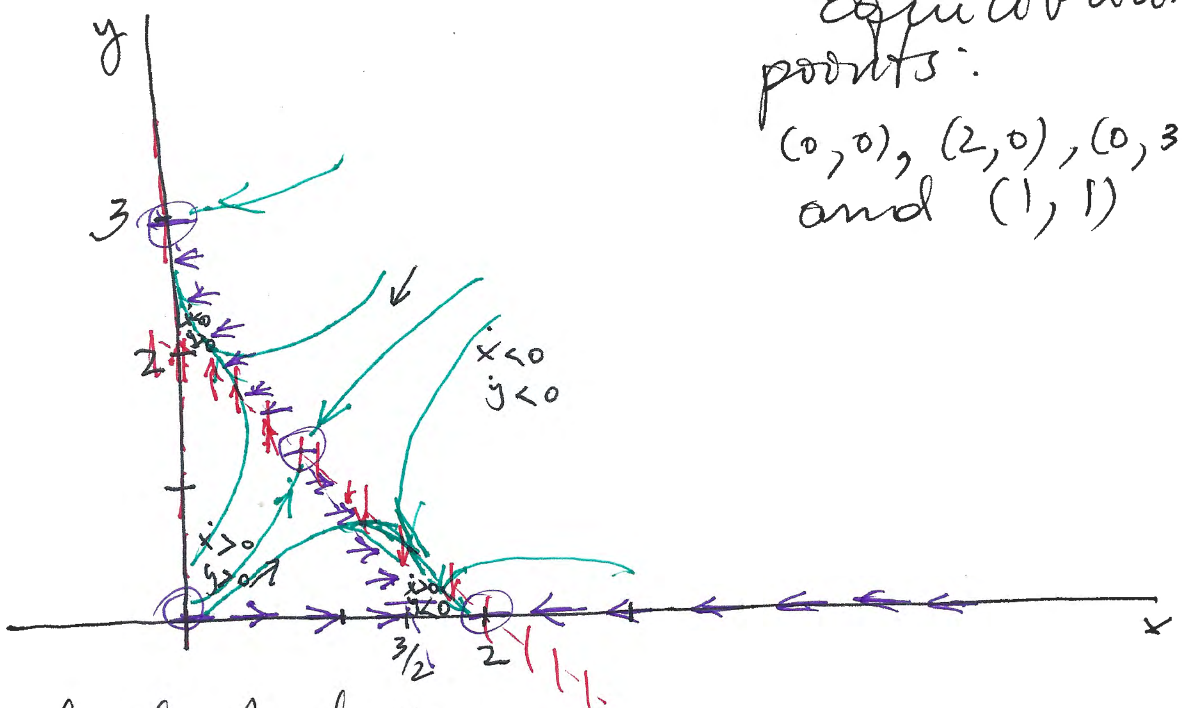
Do phase-plane analysis.

Nullclines

$\dot{x}=0$-nullclines $\to$ $x=0$ (y-axis) and $x+y=2$

$\dot{y}=0$ - nullclines $\to$ $y=0$ (x-axis) and $2x+y=3$.

These are sketched below.



Equilibrium points:

(0,0), (2,0), (0,3) and (1,1)

Local Analysis

Compute the derivative of $F(x,y) = \begin{pmatrix} 2x - x^2 - xy \\ 3y - 2xy - y^2 \end{pmatrix}$

to get

$$DF(x,y) = \begin{pmatrix} 2-2x-y & -x \\ -2y & 3-2x-2y \end{pmatrix}$$

At (0,0):

$DF(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ has two positive eigenvalues; thus, (0,0) is a source, by the Principle of Linearized Stability.

3. (Continued)

At $(2, 0)$

$$DF(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$$ has eigenvalues:

$\lambda_1 = -2$ and $\lambda_2 = -1$. Thus, $(2, 0)$ is a sink, by the Principle of Linearized Stability.

At $(0, 3)$

$$DF(0,3) = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}$$ has

eigenvalues: $\lambda_1 = -3$ and $\lambda_2 = -1$.

Thus, $(0, 3)$ is also a sink, by the Principle of Linearized Stability.

At $(1, 1)$

$$DF(1,1) = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 2\lambda - 1.$$

Thus, the eigenvalues of $DF(1,1)$ are

$\lambda_1 = -1 - \sqrt{2}$ and $\lambda_2 = \sqrt{2} - 1$

So $\lambda_1 < 0$ and $\lambda_2 > 0$. Therefore, by the Principle of Linearized Stability, $(1, 1)$ is a saddle point. Since $(1,1)$ is unstable, it is practically impossible to get to that point.

4. Given $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = 4x + 7y, \quad \text{for } (x,y) \in \mathbb{R}^2,$$

and $C = \{(x,y) \in \mathbb{R}^2 \mid 4x^2 + y^2 = 4\}$.

Then, C is a level set of the function $g: \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y) = 4x^2 + y^2, \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Consequently, $\nabla g(x,y)$ is perpendicular to C at (x,y). We want to find points $(x,y) \in C$ at which ∇f is parallel to $\nabla g(x,y)$; that is,

$$\nabla g = \lambda \nabla f,$$

for some scalar λ, where

$$\nabla g = \begin{pmatrix} 8x \\ 2y \end{pmatrix} \quad \text{and} \quad \nabla f = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 8x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ 7 \end{pmatrix},$$

from which we get the equations

$$\begin{cases} 8x = 4\lambda \\ 2y = 7\lambda, \end{cases}$$

or

$$x = \frac{\lambda}{2} \quad \text{and} \quad y = \frac{7}{2}\lambda.$$

4. (Continued)

Since the point (x, y) is on C, we also must have that

$$4\left(\frac{\lambda}{2}\right)^2 + \left(\frac{7\lambda}{2}\right)^2 = 4,$$

from which we get

$$\lambda^2 + \frac{49\lambda^2}{4} = 4,$$

or $$4\lambda^2 + 49\lambda^2 = 16,$$

or $$53\lambda^2 = 16.$$

Thus $\lambda = \pm \dfrac{4}{\sqrt{53}} = \pm \dfrac{4\sqrt{53}}{53}.$

For each value of λ, we get a point on C at which ∇f is parallel to ∇g, and therefore ~~par~~ perpendicular to C!

$$\left(\frac{2\sqrt{53}}{53}, \frac{14\sqrt{53}}{53}\right) \text{ and } \left(-\frac{2\sqrt{53}}{53}, \frac{-14\sqrt{53}}{53}\right).$$

5. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$, and define $H : D \to \mathbb{R}$ by

$$H(x,y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y),$$

for $(x,y) \in D$, where α, β, γ and δ are positive parameters involved on The Lotka-Volterra system

$$\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = \delta x y - \gamma y. \end{cases} \qquad (3)$$

(a) Compute

$$\frac{\partial H}{\partial x} = \delta - \frac{\gamma}{x}, \quad \frac{\partial H}{\partial y} = \beta - \frac{\alpha}{y}, \quad x>0, y>0$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{\gamma}{x^2}, \quad \frac{\partial^2 H}{\partial y^2} = \frac{\alpha}{y^2},$$

$$\frac{\partial^2 H}{\partial y \partial x} = 0 = \frac{\partial^2 H}{\partial x \partial y}.$$

(b) $\nabla H = \mathbf{0}$ (the zero vector) iff $H_x = 0$ and $H_y = 0$

or $\bar{x} = \frac{\gamma}{\delta}$ and $\bar{y} = \frac{\alpha}{\beta}$

Thus, $\nabla H = 0$ at $(\bar{x}, \bar{y}) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$.

5. (Continued)

(c) Let $(x(t), y(t))$ denote a solution curve of the Lotka-Volterra system on (3). Then, applying the Chain Rule,

$$\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt},$$

$$= \left(\delta - \frac{\gamma}{x}\right)[\alpha x - \beta xy] + \left(\beta - \frac{\alpha}{y}\right)[\delta xy - \gamma y]$$

$$= \alpha\delta x - \cancel{\beta\delta xy} - \cancel{\alpha\gamma} + \cancel{\beta\gamma y}$$
$$+ \cancel{\beta\delta xy} - \cancel{\beta\gamma y} - \cancel{\alpha\delta x} + \cancel{\alpha\gamma}$$

$$= 0, \text{ for all } t;$$

So that, H is constant on any solution curve of (3) on D.

(d) Solve the system $\begin{cases} \alpha x - \beta xy = 0 \\ \delta xy - \gamma y = 0 \end{cases}$

or $\begin{cases} x(\alpha - \beta y) = 0 \\ y(\delta x - \gamma) = 0 \end{cases}$. Thus, since $x > 0$ and $y > 0$ on D, we get

$$\begin{cases} \alpha - \beta y = 0; \\ \delta x - \gamma = 0, \end{cases}$$

which yields $\bar{x} = \dfrac{\gamma}{\delta}$ and $\bar{y} = \dfrac{\alpha}{\beta}$.

Thus, the only equilibrium point of (3) on D is $(\bar{x}, \bar{y}) = \left(\dfrac{\gamma}{\delta}, \dfrac{\alpha}{\beta}\right)$.

(e) Use Taylor's Theorem to write

$$H(x,y) \cong H(\bar{x},\bar{y}) + H_x(\bar{x},\bar{y})(x-\bar{x}) + H_y(\bar{x},\bar{y})(y-\bar{y})$$
$$+ \frac{1}{2}\Big[H_{xx}(\bar{x},\bar{y})(x-\bar{x})^2 + 2H_{xy}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y})$$
$$+ H_{yy}(\bar{x},\bar{y})(y-\bar{y})^2 \Big]$$
$$\cong \frac{1}{2}\frac{\gamma}{(\bar{x})^2}(x-\bar{x})^2 + \frac{1}{2}\frac{\alpha}{(\bar{y})^2}(y-\bar{y})^2$$

when (x,y) is close to $(\bar{x},\bar{y})$.

Thus, the graph of $z = H(x,y)$ near $(\bar{x},\bar{y})$ looks like the graph of the paraboloid

$$z = \frac{\gamma}{2(\bar{x})^2}(x-\bar{x})^2 + \frac{\alpha}{2(\bar{y})^2}(y-\bar{y})^2$$

which has a minimum at $(\bar{x},\bar{y})$.